1.1. **Binomial interest-rate trees.** Provide your **complete solution** to the following problem(s):

**Problem 1.1.** (10 points)

The evolution of effective annual interest rates is modeled by the following two-step binomial interest-rate model:

\[
\begin{array}{cccc}
0.04 & 0.05 & 0.035 & 0.03 \\
0.045 & 0.06 & 0.04 & 0.05 & 0.035 & 0.03 & 0.045 & 0.06
\end{array}
\]

The (risk-neutral) probability of a step up is 3/5 for every single step.

Calculate the effective yield rate of a zero-coupon bond redeemable at time $-3$.

**Solution:** The bond’s price can be calculated as

\[
P(0, 3) = (1.04)^{-1}[(1.05)^{-1}(0.36(1.06)^{-1} + 0.24(1.045)^{-1}) + (1.035)^{-1}(0.24(1.045)^{-1} + 0.16(1.03)^{-1})]
\]

\[= 0.879.\]

The bond’s yield $y$ satisfies

\[
P(0, 3)(1 + y)^3 = 1 \quad \Rightarrow \quad y = P(0, 3)^{-1/3} - 1 = 0.04393.
\]
Problem 1.2. (9 points) The evolution of effective annual interest rate over the following three years is modeled using the following binomial interest-rate tree:

The risk-neutral probability of moving up in a single step in the tree is given to be $\frac{3}{5}$.

Calculate the price of a caplet insuring the third interest payment of a 3-year, $10,000 loan with interest-only payments at the end of every year. The cap rate is given to be $0.033 = 3.3\%$.

Solution:

$$
10,000 \times \frac{1}{1.04} \times \left[ \left( \frac{3}{5} \right)^2 \times \frac{0.05 - 0.033}{(1.05)(1.045)} + \frac{3 \times 2}{5^2} \times (0.035 - 0.033) \left( \frac{1}{(1.035)(1.045)} + \frac{1}{(1.035)(1.03)} \right) \right]
$$

$$
= 62.2273.
$$

1.2. Black-Derman-Toy. Provide the final answer to the following problem(s):

Problem 1.3. (5 points) The price of a zero-coupon bond redeemable in one year equals $93.50, while the price of a zero-coupon bond redeemable in two year equals $82.50.

You are using the above bond prices to calibrate a Black-Derman-Toy tree of effective annual interest rates under the additional assumption that the volatility of interest rates in the second period equals 0.09.

Let $r_d$ denote the interest rate in the “down” state. Then, $r_d$ falls within the following interval:

(a) $[0, 0.07)$
(b) $[0.07, 0.09)$
(c) $[0.09, 0.12)$
(d) $[0.12, 0.14)$
(e) None of the above.

**Solution: (d)**

We need to solve for $r_d$ in

$$82.50 = 93.50 \times \frac{1}{2} \times \left( \frac{1}{1 + r_d e^{0.18}} + \frac{1}{1 + r_d} \right).$$

Simplifying the above equation, we get this quadratic in $r_d$

$$2.11r_d^2 + 1.68r_d − 0.24 = 0.$$ 

So, $r_d = 0.1237$. 

Provide a complete solution to the following problem(s):

**Problem 1.4.** (8 points) *Source: Problems 24.12 and 24.13 from the textbook.*

Here is an incomplete Black-Derman-Toy interest rate tree with effective annual interest rates at each node.

(i) (2 pts) Calculate $r_{ud}$.

(ii) (3 pts) What is the 3-year zero coupon bond price per $100 at maturity implied by this tree? Assume that the bond is issued at time 0.

(iii) (3 pts) What volatilities of annual effective interest rates were used to construct the above tree?

**Solution:**

(i)

$$r_{ud}^2 = r_{uu} \times r_{dd} = 0.08170 \times 0.13843 \approx 0.01131.$$ 

So, $r_{ud} \approx 0.10635$.

(ii)

$$P(0, 3) = \frac{1}{4} \left[ (1.08170 \times 1.07676 \times 1.08)^{-1} 
+ (1.10635 \times 1.07676 \times 1.08)^{-1} 
+ (1.10635 \times 1.10362 \times 1.08)^{-1} 
+ (1.13843 \times 1.10362 \times 1.08)^{-1} \right]$$

$$= \frac{1}{4} \left[ 0.794969563 + 0.777257266 + 0.758340311 + 0.736970918 \right] \approx 0.76572.$$ 

(iii)

$$\sigma_1 = \frac{1}{2} \ln \left( \frac{r_{uu}}{r_{dd}} \right) \approx 0.15.$$ 

$$\sigma_2 = \frac{1}{2} \ln \left( \frac{r_{uu}}{r_{ud}} \right) \approx 0.13.$$
Problem 1.5. (8 points) Source: Problems 24.12 and 24.13 from the McDonald textbook.
Here is an incomplete Black-Derman-Toy interest rate tree with effective annual interest rates at each node.

\[ \begin{array}{c}
0.08000 \\
0.08112 \\
0.09908 \\
0.08749 \\
0.10689 \\
0.08749
\end{array} \]

(i) (2 pts) Calculate \( r_{uu} \).
(ii) (3 pts) What is the 3-year zero coupon bond price per $100 at maturity implied by this tree? Assume that the bond is issued at time 0.
(iii) (3 pts) What volatilities of annual effective interest rates were used to construct the above tree?

Solution:
(i) 
\[ r_{uu} = \frac{r_{ud}^2}{r_{dd}} = \frac{0.10689^2}{0.08749} = 0.130592. \]

(ii) 
\[ B(0, 3) = \frac{1}{4}[ (1.08749 \times 1.08112 \times 1.08)^{-1} \\
+ (1.10689 \times 1.08112 \times 1.08)^{-1} \\
+ (1.10689 \times 1.09908 \times 1.08)^{-1} \\
+ (1.130592 \times 1.09908 \times 1.08)^{-1} ] \\
= \frac{1}{4} [0.787548 + 0.773745 + 0.761101 + 0.745145] \approx 0.766885. \]
So, per $100, the price is 76.6885.

(iii) 
\[ \sigma_1 = \frac{1}{2} \ln\left( \frac{r_u}{r_d} \right) \approx 0.10. \]
\[ \sigma_2 = \frac{1}{2} \ln\left( \frac{r_{uu}}{r_{ddu}} \right) \approx 0.10. \]

Problem 1.6. (10 points)
Consider the following values of interest rates from an incomplete Black-Derman-Toy interest rate tree for the effective annual interest rates.
Calculate the yield volatility in year one of a zero-coupon bond with maturity three years from today.

**Solution:** First, we calculate the lowest of the three possible effective interest rates after two steps:

\[
r_{dd} = \frac{r_{ud}^2}{r_{uu}} = \frac{0.15^2}{0.19} = 0.1184.
\]

The bond prices at the up and down nodes are

\[
P_u = \frac{1}{1.15} \times \frac{1}{2} \times \left( \frac{1}{1.19} + \frac{1}{1.15} \right) = 0.743435,
\]

\[
P_d = \frac{1}{1.13} \times \frac{1}{2} \times \left( \frac{1}{1.15} + \frac{1}{1.1184} \right) = 0.780398.
\]

The bond yields at the up and down nodes are

\[
y_u = P_u^{-1/2} - 1 = 0.159788, \quad y_d = P_d^{-1/2} - 1 = 0.131988.
\]

So, the required yield volatility equals

\[
\kappa = \frac{1}{2} \ln \left( \frac{y_u}{y_d} \right) = 0.0956.
\]