1.1. **Prerequisite material.** Provide your final answer only for the following problems.

**Problem 1.1.** (5 pts) *Source: Problem 16a (p.331) from Kellison.*
You are given the following table of spot rates:

<table>
<thead>
<tr>
<th>Length of Investment</th>
<th>Spot rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 year</td>
<td>0.0700</td>
</tr>
<tr>
<td>2 years</td>
<td>0.0800</td>
</tr>
<tr>
<td>3 years</td>
<td>0.0875</td>
</tr>
<tr>
<td>4 years</td>
<td>0.0925</td>
</tr>
<tr>
<td>5 years</td>
<td>0.0950</td>
</tr>
</tbody>
</table>

Find the price of a $1,000 three-year bond with annual 5% coupons.

(a) About 800
(b) About 850
(c) About 900
(d) About 950
(e) None of the above

**Solution:** (c)

\[
50 \cdot (1 + 0.07)^{-1} + 50 \cdot (1 + 0.08)^{-2} + 1050 \cdot (1 + 0.0875)^{-3} \approx 905.99.
\]

**Problem 1.2.** (5 pts) Consider a non-dividend-paying stock currently priced at $100 per share. The price of this stock in one year is modeled using a one-period binomial tree under the assumption that the stock price can either go up to 110 or down to 90.

Let the continuously compounded, risk-free interest rate equal 0.04. What is the risk-neutral probability of the stock price going up?

**Solution:**

\[
p^* = \frac{100e^{0.04} - 90}{110 - 90} = 0.7041.
\]

1.2. **Forwards on bonds.**

**Problem 1.3.** (5 points) The spot rates for zero-coupon bonds are observed to be 

\[
r_0(0, 1) = 0.03, \quad r_0(0, 2) = 0.04, \quad r_0(0, 3) = 0.045, \quad r_0(0, 4) = 0.0475.
\]

What is the forward price for delivery in one year of a zero-coupon bond with three years left to maturity? Assume the redemption amount of $100.

(a) About $85.55
(b) About $90.26
(c) About $98.33
(d) About $98.56
(e) None of the above.

**Solution:** (a)

\[
F_{0,1}[P(1, 4)] = \frac{P(0, 4)}{P(0, 1)} = \frac{1 + r_0(0, 1)}{(1 + r_0(0, 4))^4} = 0.8555.
\]
1.3. Binomial interest-rate trees. Provide your **complete solution** to the following problem(s):

**Problem 1.4.** (10 points) The evolution of effective annual interest rates is modeled by a three-period tree so that during each period the interest rate can either increase by 0.02 or decrease by 0.01. Let the initial (root) interest rate be equal to 0.08. The risk-neutral probability of an up movement is 0.6 at every node.

Consider a three-year interest-rate cap with a cap rate equal to 0.09 purchased to hedge against adverse movements of the above-modeled floating rate. Let the loan amount be equal to $1000 and let the loan repayment installments be interest-only.

What is the price of the above cap consistent with our model?

**Solution:** The nodes in the tree at which the cap makes a positive payment, along with the respective payment amounts are:

\[\text{up} : 1000(0.10 - 0.09) = 1000(0.01),\]
\[\text{up - up} : 1000(0.12 - 0.09) = 1000(0.03).\]

So, the cap’s price equals

\[
1000 \times \frac{1}{1.08} \left[0.6 \times \frac{0.01}{1.1} + (0.6)^2 \times \frac{0.03}{1.12(1.1)}\right] = \frac{1000(0.6)(0.01)}{1.08(1.1)} \left[1 + \frac{0.6(3)}{1.12}\right] = \frac{6}{1.08(1.1)} (2.607143) = 13.16739
\]

**Problem 1.5.** (10 points)

The evolution of **effective** annual interest rates is modeled by the following two-step binomial interest-rate model:
The (risk-neutral) probability of a step up is $3/5$ for every single step. Calculate the effective yield rate of a zero-coupon bond redeemable at time $-3$.

Solution: The bond’s price can be calculated as

$$P(0, 3) = (1.04)^{-1} \left[ (1.05)^{-1}(0.36(1.06)^{-1} + 0.24(1.045)^{-1}) + (1.035)^{-1}(0.24(1.045)^{-1} + 0.16(1.03)^{-1}) \right]$$

$$= 0.879.$$

The bond’s yield $y$ satisfies

$$P(0, 3)(1 + y)^3 = 1 \quad \Rightarrow \quad y = P(0, 3)^{-1/3} - 1 = 0.04393.$$
Problem 1.6. (10 points) The evolution of effective annual interest rate over the following three years is modeled using the following binomial interest-rate tree:

![Interest-rate Tree Diagram]

The risk-neutral probability of moving up in a single step in the tree is given to be 3/5. Calculate the price of a caplet insuring the third interest payment of a 3-year, $10,000 loan with interest-only payments at the end of every year. The cap rate is given to be 0.033 = 3.3%.

Solution:

\[
10,000 \times \frac{1}{1.04} \times \left[ \left( \frac{3}{5} \right)^2 \times \frac{0.05 - 0.033}{(1.05)(1.045)} + \left( \frac{3 \times 2}{5^2} \right) \times (0.035 - 0.033) \left( \frac{1}{(1.035)(1.045)} + \frac{1}{(1.035)(1.03)} \right) \right] \\
= 62.2273.
\]

Problem 1.7. (5 points) The two-period interest-rate tree models effective annual interest rates. The two possible interest rates for the time period [1, 2] are given to be 0.04 and 0.02. The risk-neutral probability of an up move is specified as 0.55.

The price of a zero-coupon bond redeemable in two years for $1 is $0.95.

What is the root effective interest rate consistent with the above bond prices?

(a) About 0.021.
(b) About 0.023.
(c) About 0.024.
(d) About 0.025.
(e) None of the above.
Solution: (a)

\[ 0.95 = P(0, 1) \left[ \frac{0.55}{1.04} + \frac{0.45}{1.02} \right] \Rightarrow r_0 = 0.021. \]