2.1. **Subjective probabilities.** Provide your **complete solution** to the following problem(s):

**Problem 2.1.** (10 points)

The current price of a non-dividend-paying stock is $40 per share. The stock’s volatility is given to be 0.30. The stock’s price in one year is modeled using the Cox-Ross-Rubinstein model, i.e., the up and down factors are given to be
\[ u = e^{\sigma \sqrt{h}}, \quad d = e^{-\sigma \sqrt{h}}, \]
with \( h = 1 \). The continuously compounded risk-free interest rate is 5%.

What is the mean rate of return of an investment made in this stock under the real probability of an up movement given to be \( \frac{2}{3} \)?

**Solution:** As we learned in class, the mean rate of return in this case can be written as
\[ \alpha = \ln(pu + (1 - p)d). \]

So, in this problem,
\[ \alpha = \ln \left( \frac{2}{3} e^{0.3} + \frac{1}{3} e^{-0.3} \right) = 0.1370. \]

**Problem 2.2.** The current price of a continuous-dividend-paying stock is given to be 80. Its dividend yield is given to be 0.04 and its volatility is given to be 0.25. The stock price evolution over the following half year is modeled using a two-period forward binomial tree. The continuously compounded risk-free interest rate is 0.06.

We anticipate that the probability of an up movement is 0.55 at every node.

(i) (10 points) What is the expected price half a year from now consistent with the above model?

(ii) (5 points) What is the annualized continuously compounded rate of return consistent with the above subjective probability?

(iii) (10 points) Based on the above model, would you rather long a forward contract or short a forward contract?

(iv) (10 points) Find a value of the subjective probability such that your decision would be contrary to the one you made in the previous part of the problem.

**Solution:**

(i) The three possible stock prices are
\[ S_{uu} = S(0)u^2 = 80e^{2[(0.06-0.04)(0.25)+0.25\sqrt{0.25}]} = 80e^{0.26}, \]
\[ S_{ud} = S(0)ud = 80e^{2[(0.06-0.04)(0.25)]} = 80e^{0.01}, \]
\[ S_{dd} = S(0)d^2 = 80e^{2[(0.06-0.04)(0.25)-0.25\sqrt{0.25}]} = 80e^{-0.24}. \]

So, the expected stock price consistent with the subjective probability of an up movement 0.55 in a single step equals
\[ \mathbb{E}[S(1/2)] = 80e^{-0.24} \left[ (0.55)^2 e^{0.50} + 2(0.55)(0.45)e^{0.25} + (0.45)^2 \right] = 80e^{-0.24}(0.55e^{0.25} + 0.45)^2 = 84.12707. \]

(ii) The quantity we seek, denoted by \( \alpha \), must satisfy
\[ 80e^{0.5(\alpha-\delta)} = 84.12707 \implies \alpha = 2 \ln(1.051588) + 0.04 = 0.1406. \]

(iii) If we long a forward contract, the forward price will be
\[ F_{0,1/2}(S) = 80e^{(0.06-0.04)(0.5)} = 80e^{0.01} = 80.80401. \]

Since we expect that the stock price will be 84.12707, we should long the forward contract.
(iv) Consider a model in which the annualized continuously compounded rate of return equals $\tilde{\alpha} = 0.05$. Then, the expected value of the stock price consistent with this return equals

$$\tilde{E}[S(1/2)] = 80e^{0.005} < 80e^{0.01}.$$ 

The subjective probability consistent with the above return is

$$\tilde{p} = \frac{e^{(\tilde{\alpha}-\delta)h} - d}{u - d} = 0.4689.$$ 

**Problem 2.3.** (5 points)

Let $\{X_k, k = 1, 2, \ldots\}$ be a sequence of independent, identically distributed random variables such that $\mathbb{E}[X_1] = 1$ and $\text{Var}[X_1] = 2$. Find

(i) (2 points)

$$\lim_{n \to \infty} \frac{X_1 + X_2 + \cdots + X_n}{n}$$

(ii) (3 points)

$$\lim_{n \to \infty} \frac{X_1^2 + X_2^2 + \cdots + X_n^2}{n}$$

**Solution:** Using the law of large numbers, we get

(i)

$$\lim_{n \to \infty} \frac{X_1 + X_2 + \cdots + X_n}{n} = \mathbb{E}[X_1] = 1$$

(ii)

$$\lim_{n \to \infty} \frac{X_1^2 + X_2^2 + \cdots + X_n^2}{n} = \mathbb{E}[X_1^2] = \text{Var}[X_1] + (\mathbb{E}[X_1])^2 = 2 + 1 = 3.$$