4.1. **The Black-Scholes pricing formula.** Please, provide your complete solution to the following problem(s):

**Problem 4.1.** (7 points) Solve Sample MFE Problem #33.

**Solution:** (c)

**Problem 4.2.** *Source: McDonald.*

Suppose that $ABC$ is a non-dividend-paying stock whose price is modeled using the Black-Scholes pricing assumptions. Assume that its spot price is $100, the volatility is 0.4. Let $r = 0.06$.

- a. (5 pts) What is the price of a European call option with 105 strike and 1 year to expiration?
- b. (5 pts) What is the 1-year forward price on the $ABC$ stock?
- c. (5 pts) What is the price of a European call option with strike 105 and maturity 1 year for which the underlying asset is a *futures contact* whose maturity is the same as that of the option, i.e., 1 year?

**Solution:**

- a. Using the Black-Scholes formula, we find a call-price of $16.33$.
- b. We determine the one year forward price as follows:

$$F_{0,T}(S) = Se^{rT} = 100e^{0.06 \cdot 1} = 106.1837$$

- c. By the Black formula, we simply need to modify the Black-Scholes formula so that the role of the dividend yield in the Black-Scholes formula is played by the risk-free rate, while the role of the spot-price is played by the forward price evaluated in part b. Thus,

$$V_C(106.1837, 105, 0.4, 0.06, 1, 0.06) = 16.33$$

This exercise shows the general result that a European futures option has the same value as the European stock option provided the futures contract has the same expiration as the stock option.

**Problem 4.3.** *Source: Problem #12.8 from McDonald.*

In the Black-Scholes setting, assume that $S(0) = 100, r = 0.08, \sigma = 0.3, \delta = 0.03$.

- a. (5 pts) What is the 9-month forward price for the stock $S$?
- b. (8 pts) Compute the price of a 95−strike, 9−month call option on a futures contract in $S$.

**Solution:**

- a. We have to be careful here: We have to take into account the dividend yield when calculating the 9-month forward price.

$$F_{0,T}(S) = S(0)^{e^{(r-\delta)T}} = 100e^{(0.08 - 0.03)0.75} = 103.8212.$$ 

- b. Having found the correct forward price, we can use the Black-Scholes prising formula for futures options:

$$V_C(103.8212, 95, 0.3, 0.08, 0.75, 0.08) = 14.3863.$$ 

**Problem 4.4.** (5 points) Assume the Black-Scholes setting.

Suppose the spot exchange rate is $1.43 per British pound and the strike on a dollar denominated pound call is $1.30. Assume that the USD risk-free rate is $r = 0.045$, and that the GBP risk-free rate is $r_f = 0.06$. Let $\sigma = 0.15$ and let the option expire in 180 days (simplify the number of days in a year to 360).

What is this European call option’s price?
Solution: In our usual notation, the price is
\[ V_C(0) = X(0)e^{-r_f T}N(d_1) - Ke^{-r_E T}N(d_2) \]
with
\[ d_1 = 0.88, \quad d_2 = 0.77. \]
So, \( V_C(0) = 0.134. \)

Problem 4.5. (10 points) Source: Problem 14.13 from McDonald.
Consider a gap put option in US dollars on the euro. Denote the dollar per euro exchange rate at time \( t \) by \( x(t) \) and assume the Black-Scholes model applies.
Set \( x(0) = 0.9, \sigma = 0.1, r_s = 0.06, r_E = 0.03 \). Let the gap put option have maturity \( T = 0.5 \), strike price \( K_1 = 0.8 \) and the payment trigger \( K_2 = 1 \).

(i) (5 points) Find the price of the above gap put option.

(ii) (5 points) If the volatility were zero, and you kept all of the other parameters as above, what would the price of this gap put option be?

Solution:

(i) In the usual notation, the expression for the price of the gap put is
\[ V_{GP}(0) = K_1e^{-r_s T}N(-d_2) - x(0)e^{-r_E T}N(-d_1) \]
with
\[ d_1 = \frac{1}{\sigma \sqrt{T}} \left[ \ln \left( \frac{x(0)}{K_2} \right) + (r_s - r_E + \frac{1}{2}\sigma^2)T \right] = -1.2425, \]
\[ d_2 = d_1 - \sigma \sqrt{T} = -1.3133. \]
with \( K_1 \) the strike and \( K_2 \) the payment trigger.

The gap-put price turns out to be
\[ V_{GP}(0) = -0.0887. \]

The interpretation is that the cashflow will move from the writer of the option to the “buyer”.

(ii) A gap put will pay \( 0.8 - x \) when \( x < 1 \). With zero volatility, \( x = x_0 = .9 \) and this will mean we will be selling the foreign currency for .8 dollars. This is equivalent to a forward contract with delivery price \( K_1 = 0.8 \),
\[ -0.9e^{-0.03/2} + 0.8e^{-0.06/2} = -0.11024. \] (4.1)

Or, one can use the pricing formula from part (i). It is not allowed to simply plug in \( \sigma = 0 \) into \( d_1 \) and \( d_2 \) since that would mean dividing by 0. But, you can let \( \sigma \to 0 \) in that expression.