**Problem 5.1.** (5 points) Assume the Black-Scholes model. Consider the stock-price process \( S = \{ S(t); t \geq 0 \} \) for a continuous-dividend-paying stock under the risk-neutral probability measure. Denote the risk-free continuously compounded interest rate by \( r \).

Calculate \( E^*[e^{-rt}S(t)] \) for every \( t \geq 0 \) where \( E^* \) stands for the expectation under the risk-neutral probability measure.

**Solution:**

\[
E^*[Y(t)] = E^*[e^{-rt}S(t)] = e^{-rt}E^*[S(t)] = e^{-rt}S(0)e^{(r-\delta)t} = S(0)e^{-\delta t}.
\]

**Problem 5.2.** (5 points) Assume the Black-Scholes setting.

Assume \( S(0) = 23.50 \), \( \sigma = 0.24 \), \( r = 0.055 \). The stock pays a 2.5% continuous dividend and the option expires in 45 days (simplify the number of days in a year to 360).

What is the price of a $25-strike European call?

**Solution:** In our usual notation, the price is

\[
V_C(0) = S(0)e^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)
\]

with

\[
d_1 = -0.64, \quad d_2 = -0.73.
\]

So, \( V_C(0) \approx 0.3 \).

**Problem 5.3.** (10 pts) In this problem, use the Black-Scholes pricing model.

Consider a bear spread consisting of a 20-strike put and a 25-strike put. Suppose that \( \sigma = 0.30, r = 0.04, \delta = 0.02, T = 1 \) and \( S(0) = 5 \).

What is the price of this bear spread?

**Solution:** Here, you sell a 20-strike put and buy a 25-strike put.

For the 20-put, we have

\[
d_1 = \frac{1}{0.3 \cdot 1} \left[ \ln \left( \frac{5}{20} \right) + (0.04 - 0.02 + \frac{1}{2} \cdot 0.3^2) \cdot 1 \right] = \frac{1}{0.3} \left[ \ln(0.25) + 0.065 \right] = -4.4043;
\]

\[
d_2 = -4.4043 - 0.3 = -4.7043.
\]

For the 25-put, we have

\[
d_1 = \frac{1}{0.3 \cdot 1} \left[ \ln \left( \frac{5}{25} \right) + (0.04 - 0.02 + \frac{1}{2} \cdot 0.3^2) \cdot 1 \right] = \frac{1}{0.3} \left[ \ln(0.2) + 0.065 \right] = -5.1481;
\]

\[
d_2 = -5.1481 - 0.3 = -5.4181.
\]

We conclude that for both options, we have

\[
N(-d_1) \approx N(-d_2) = 1.
\]

We calculate the prices of the two puts:

\[
V_P(0, 20) \approx 20e^{-0.04} - 5e^{-0.02} = 14.3148
\]

\[
V_P(0, 25) \approx 25e^{-0.04} - 5e^{-0.02} = 19.1187.
\]

So, the price of the bear spread is $4.8039.

Or, simply, the price of the bear spread is

\[
V_P(0, 25) - V_P(0, 20) \approx 25e^{-0.04} - 5e^{-0.02} - (20e^{-0.04} - 5e^{-0.02}) = 5e^{-0.04} \approx 4.8039.
\]

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Problem 5.4. Let \( S(0) = 100 \), \( K = 120 \), \( \sigma = 0.3 \), \( r = 0.08 \) and \( \delta = 0 \).

a. (10 pts) Let \( V_C(0, T) \) denote the Black-Scholes European call price for the maturity \( T \). Does the limit of \( V_C(0, T) \) as \( T \to \infty \) exist? If it does, what is it?

b. (10 pts) Now, set \( \delta = 0.001 \) and let \( V_C(0, T, \delta) \) denote the Black-Scholes European call price for the maturity \( T \). Again, how does \( V_C(0, T, \delta) \) behave as \( T \to \infty \)?

c. (5 pts) Interpret in a sentence or two the differences, if there are any, between your answers to questions in a. and b.

Solution:

a. By the Black-Scholes pricing formula, the function \( V_C(0, T) \) has the form
\[
V_C(0, T) = S(0)N(d_1) - Ke^{-rT}N(d_2),
\]
where \( N \) denotes the distribution function of the unit normal distribution and
\[
d_1 = \frac{1}{\sigma\sqrt{T}} \left[ \ln \left( \frac{S(0)}{K} \right) + \left( r + \frac{1}{2} \sigma^2 \right) T \right],
\]
\[
d_2 = d_1 - \sigma\sqrt{T}.
\]
As \( T \to \infty \), we have that
\[
d_1 \to \infty \Rightarrow N(d_1) \to 1,
\]
\[
e^{-rT}N(d_2) \leq e^{-rT} \to 0.
\]
Hence,
\[
V_C(0, T) \to S(0), \text{ as } T \to \infty.
\]

b. In this case, the price of the call option reads as
\[
V_C(0, T, \delta) = S(0)e^{-\delta T}N(d_1) - Ke^{-rT}N(d_2),
\]
with
\[
d_1 = \frac{1}{\sigma\sqrt{T}} \left[ \ln \left( \frac{S(0)}{K} \right) + \left( r - \delta + \frac{1}{2} \sigma^2 \right) T \right],
\]
\[
d_2 = d_1 - \sigma\sqrt{T}.
\]
Since the function \( N \) is bounded between 0 and 1, we see that as \( T \to \infty \), \( V_C(0, T, \delta) \to 0 \).

c. When the stock is paying the dividend, the benefit of owning the stock and opposed to owning the option on that stock lies precisely in the value of the issued dividend. As we can see from above, even a very small dividend yield is going to render the call options for very long maturities worthless.

Problem 5.5. (5 points) Source: Sample MFE Problem #41.
The stock price \( S = \{ S(t), t \geq 0 \} \) is modeled by a lognormal distribution for every \( t \). The initial stock price is given to be 45 while the stock’s volatility equals 0.25. The stock pays dividends continuously at a rate proportional to its price and with the dividend yield equal to 0.03.

The continuously compounded risk-free interest rate is 0.07.
Consider the one-year European contingent-claim on the above stock with the payoff function
\[
v(s) = \min(s, 42).
\]
Find the time–0 price \( V(0) \) of the above contingent claim.

Solution: The payoff of the above contingent claim can be rewritten as
\[
v(s) = \min(s, 42) = s + \min(0, 42 - s) = s - \max(0, s - 42) = s - (s - 42)_+.
\]
We conclude that the contingent claim can be replicated using a portfolio with the following two components:
- one long prepaid forward contract on the stock with delivery date in one year, and
- one written one-year, 42-strike European call option on the stock.
So,
\[ V(0) = F_{0,1}(S) - V_C(0) = 45e^{-0.03} - V_C(0) \]

with
\[ V_C(0) = S(0)e^{-\delta S T} N(d_1) - Ke^{-r T} N(d_2) \]

where
\[ d_1 = \frac{1}{\sigma \sqrt{T}} \left[ \ln \left( \frac{S(0)}{K} \right) + (r - \delta + \frac{1}{2} \sigma^2) T \right] = 0.56, \]
\[ d_2 = d_1 - \sigma \sqrt{T} = 0.31. \]

We get \( V_C(0) = 4.94 \). So, the final answer is
\[ V(0) = 45e^{-0.03} - 4.94 = 36.91. \]