Binomial interest-rate trees: More periods.

So far: one-step, i.e., two-period tree

\[ r_0 \xrightarrow{} r_u \quad r_d \]

0 \quad 1 \quad 2

Remember: Stock-price trees:
\begin{align*}
S_u &= u \cdot S(0) \\
S_d &= d \cdot S(0) \quad S_{uu} &= u \cdot S_u = u^2 \cdot S(0) \\
S_{ud} &= d \cdot S_u = u \cdot S_d = u \cdot d \cdot S(0) \\
S_{dd} &= d^2 \cdot S(0)
\end{align*}

Interest-rate trees:

two-step, i.e., three-period tree

\[ r_0 \xrightarrow{P} r_u \xrightarrow{P} r_{uu} \quad r_{ud} \xrightarrow{P} r_{du} \xrightarrow{P} r_{dd} \]

\[ \text{NOT NECESSARILY EQUAL!} \]

\[ \begin{array}{ccc}
\text{Step 0} & \text{Step 1} & \text{Step 2} \\
\$1 & \text{NOT NECESSARILY EQUAL!} & \$1 \\
\end{array} \]
\[ P(0,3) = P(0,1) \left[ \frac{p^2}{1+r_{uu}} \cdot \frac{1}{1+r_u} + \right. \]

\[ + \ p(1-p) \cdot \frac{1}{1+r_{ud}} \cdot \frac{1}{1+r_u} + \]

\[ + \ (1-p) \cdot p \cdot \frac{1}{1+r_{du}} \cdot \frac{1}{1+r_d} + \]

\[ + \ (1-p)^2 \cdot \frac{1}{1+r_{dd}} \cdot \frac{1}{1+r_d} \right] \]
Assume that the binomial interest-rate tree is populated with the following effective annual interest rates:

\[ r_0 = 0.04, r_u = 0.045, r_d = 0.035. \]

We observe that a zero-coupon bond redeemable in two-years for $100 is priced at $P(0) = 92.5$ using the above interest-rate model. What is the risk-neutral probability $p$ stipulated by the model?

\[ \begin{align*}
    \Rightarrow P(0, 2) &= 0.925 \\
    &= \frac{1}{1+r_0} \left[ p \cdot \frac{1}{1+r_u} + (1-p) \cdot \frac{1}{1+r_d} \right] \\
    p^2 &= \frac{P(0, 2)(1+r_0) - P_d}{P_u - P_d} \\
    P_u &= (1.045)^{-1} = 0.956938; \\
    P_d &= (1.035)^{-1} = 0.966184. \\
    \Rightarrow p &= \frac{0.925 \cdot (1.04) - 0.966184}{0.956938 - 0.966184} \approx 0.4525. 
\end{align*} \]
Problem 1.2.  

(i) The interest rates in the binomial tree are:

\[ r_0 = 6\%, \quad r_u = 7.704\%, \quad r_{uu} = 9.892\%, \]
\[ r_d = 4.673\%, \quad r_{ud} = r_{du} = 6\%, \]
\[ r_{dd} = 3.639\%. \]

(ii) All interest rates are annual effective rates.

(iii) The risk-neutral probability that the annual effective interest rate moves up or down is \(1/2\). Find the price of a zero-coupon bond redeemable for $1,000 in three years.

\[ P(0,3) = \frac{1}{4} \left[ \frac{1}{1.09892} \cdot \frac{1}{1.07704} + \frac{1}{1.06} \cdot \frac{1}{1.07704} + \frac{1}{1.06} \cdot \frac{1}{1.04673} + \frac{1}{1.03639} \cdot \frac{1}{1.04673} \right] \]

\[ = \frac{835}{888.83} = 0.942 \]

Just for laughs: @ What's the 3-yr spot rate, i.e., the yield rate of this bond?

\[ 1000 = \frac{835.83}{(1+y)^3} \]

\[ (1+y)^{-3} = 0.83583 \Rightarrow 1+y = 0.83583^{\frac{1}{3}} \]
Continuously compounded interest rates in the tree.

\[ P_{uu} = e^{-r_{uu}t} = e^{-r_{uu}} \]

\[ P_{ud} = e^{-r_{ud}t} = e^{-r_{ud}} \]

\[ P_{du} = e^{-r_{du}t} = e^{-r_{du}} \]

\[ P_{dd} = e^{-r_{dd}t} = e^{-r_{dd}} \]

\[ P(0,3) = e^{-r_0} \left[ p^2 \cdot e^{-(r_{uu} + r_{ud})} + p(1-p) e^{-(r_{ud} + r_{du})} + (1-p)^2 \cdot p e^{-(r_{dd} + r_{du})} \right] \]
Problem 1.3. Call on a bond.

Consider a two-period binomial interest-rate tree. Assume that the root effective annual interest rate equals \( r_0 = 0.05 \). From then on, the interest rates change according to an up factor equal to \( u = 1.2 \) and the down factor \( d = 0.8 \).

Find the price of a zero-coupon bond redeemable for $1 in two years. \( \leftarrow \) Easy!

Next, price a call option on the above bond with exercise date at time \( T_o = 1 \) and with strike \( K = 0.95 \).

\[
\begin{align*}
    r_u &= 0.06 \\
    P_u &= (1.06)^{-1} = 0.943396 \\
    V_u &= 0 \\
    r_d &= 0.04 \\
    P_d &= (1.04)^{-1} = 0.961538 \\
    V_d &= (P_d - K)_+ = 0.011538 \\
    V(0) &= \frac{1}{1.05} \cdot \frac{1}{2} \cdot 0.011538 = 0.0054942
\end{align*}
\]