Problem 1.4. Getting $p$ using existing bond prices.
Assume that the binomial interest-rate tree is populated with the following effective annual interest rates:

$$r_0 = 0.04, r_u = 0.045, r_d = 0.035.$$ 

We observe that a zero-coupon bond redeemable in two-years for $100 is priced at $P(0) = 892.5$ using the above interest-rate model. What is the risk-neutral probability $p$ stipulated by the model?

\[
\begin{align*}
0.04 &= r_0 \\
0.045 &= r_u \\
0.035 &= r_d \\
\end{align*}
\]

\[
\begin{align*}
p &= \frac{1}{0.04} \left( p \left( \frac{1}{1.045} \right) + (1-p) \left( \frac{1}{1.035} \right) \right) \\
p &= 0.4525
\end{align*}
\]
Problem 9 from Spring 2007.

$r_u = 0.07704$
$r_d = 0.04673$
$r_{uu} = 0.09892$
$r_{ud} = 0.06$
$r_{dd} = 0.03639$

$a$: What is the price of a bond redeemable for $1,000 at time-3 and w/ no coupons?

\[
\frac{1}{1.06} \left( \frac{1}{1.09892} \cdot \frac{1}{1.07704} + \frac{1}{1.06} \cdot \frac{1}{1.07704} + \frac{1}{1.06} \cdot \frac{1}{1.04673} + \frac{1}{1.03639} \cdot \frac{1}{1.04673} \right) \cdot (\frac{1}{2})^2 \cdot 1000 = 835.83
\]
Forwards on bonds.

\[ 0 \quad T_D \quad T_B \]

\( T_D \) ... the delivery date for the forward contract on bond

\( T_B \) ... the maturity date of the underlying zero-coupon bond.

The mechanics:

- @ time \( -0 \): agreement on
  - the bond
  - the delivery date \( T_D \)
  - the forward price

- @ time \( T_D \): delivery of a zero-coupon bond w/ \( T_B - T_D \) left to maturity is taken
  - the forward price is paid

\[ F_{0,T_D} \left[ P(T_D,T_B) \right] = ? \]

Inputs: Prices of bonds w/ varying maturities,

i.e., for a tree w/ \( n \)-steps\([(n+1)\)-periods]\)

w/ have \( P(0,k) \) for \( k = 1,...,n+1 \)
We want to express

$$F_{0,T_D} \left[ P(T_D, T_B) \right]$$

in terms of available zero-coupon bond prices:

With $F^p$ being the prepaid forward price:

$$F^p_{0,T_D} = PV_{0,T_D}(F)$$

In the current case:

Prepaid forward contract on bonds:

$$\begin{array}{ccc}
0 & T_B & T_B \\
\uparrow F^p & & \downarrow \text{bond maturity}
\end{array}$$

$$\Rightarrow$$ Prepaid forward price on the bond equals the bond price $P(0, T_B)$ [compare to non-dividend-paying stocks!].

$$P(0, T_B) = PV_{0,T_B} \left( F_{0,T_D} \left[ P(T_D, T_B) \right] \right)$$

$$P(0, T_B) \cdot P(0, T_D)$$

$$F_{0,T_D} \left[ P(T_B, T_B) \right] = \frac{P(0, T_B)}{P(0, T_D)}$$