A “True” Probability Measure.

3.1. Pricing by Replication. So far we have used pricing by replication to figure out the price of derivatives under a binomial stock-price model. The model consisted of two constants: the up factor \( u \) and the down factor \( d \). However, a person might object to the omission of a kind of subjective probability that the stock will move up or down. Consider the following question, focusing on a simple one-period binomial model:

Let \( S \) and \( \tilde{S} \) be two stock prices modeled by the one-period binomial model with the “true historic” probabilities of the price going up equal to \( p \) and \( \tilde{p} \) respectively. Assume that \( S(0) = \tilde{S}(0) \) and that the “up” and “down” factors in both models are the same. Neither of the two stocks pays any dividends. Consider two European puts with the same strike and the same maturity on \( S \) and \( \tilde{S} \) whose prices are denoted by \( P \) and \( \tilde{P} \).

Assume that the probability of an up movement for the stock \( S \) is greater than the probability of the stock \( \tilde{S} \) going up, i.e., \( p > \tilde{p} \). What, if anything, can one conclude about \( P \) and \( \tilde{P} \)?

Due to the fact that the put option has the same payoff for both stocks, while \( \tilde{S} \) is more likely to have the “down” stock price at exercise, we might be tempted to conclude that \( \tilde{P} > P \).

On the other hand, the no-arbitrage price we already established does not use the “true” probabilities at all. So, we conclude that \( P = \tilde{P} \). In fact, it is really straightforward to find an arbitrage portfolio if this equality is not true. Is this situation paradoxical in any way?

3.2. Expected Option Payoff. Our reasoning when we decide to prefer the put on \( \tilde{S} \) to the put on \( S \) appears to involve comparing expected put payoffs:

\[
p(K - S_u) + (1 - p)(K - S_d) < \tilde{p}(K - \tilde{S}_u) + (1 - \tilde{p})(K - \tilde{S}_d) \]

\[
= \tilde{p}(K - S_u) + (1 - \tilde{p})(K - S_d).
\]

However, we should be thinking about comparing the values of the two options at time 0. To do this, we should compare the “present values” of the two payoffs. Simply discounting the two expected payoffs does not do the trick, because it does not take into account that the rate of appreciation of the put options is not the same as that of a risk-less investment. It is correct to discount a riskless investment simply using \( r \), but the same does not apply to option payoffs.

To explore this a bit further, let us start with something simpler: the expected return of stocks.

3.3. Expected Return on the Stock. Imagine that you invested \( S(0) \) dollars at the risk-free continuously compounded interest rate \( r \). Then, at the one of a single one-year period, you are able to withdraw \( S(0)e^r \).
What if you decided to invest that one single dollar in the stock \( S \), instead? For \( S(0) \) dollars, one can purchase one share of stock. So, at the end of the period, the wealth accrued is

\[
S_u \quad \text{with probability } p, \\
S_d \quad \text{with probability } 1 - p.
\]

The expected wealth is

\[
pS_u + (1 - p)S_d = puS(0) + (1 - p)dS(0) = S(0)[pu + (1 - p)d].
\]

The “mean rate of appreciation” for your stock investment is \( \alpha \) which satisfies

\[
S(0)e^\alpha = S(0)[pu + (1 - p)d] \quad \Rightarrow \quad \alpha = \ln(pu + (1 - p)d).
\]

Equivalently,

\[
p = \frac{e^\alpha - d}{u - d}.
\]

At this point, the reader should try to modify the reasoning above to the case of continuous-dividend-paying stock and a binomial tree with the period length different from one year.

**Example 3.1. MFE Spring 2007: Problem #2**

For a one-period binomial model for the price of a stock, you are given:

(i) The period is one year.
(ii) The stock pays no dividends.
(iii) \( u = 1.433 \), where \( u \) is one plus the rate of capital gain on the stock if the price goes up.
(iv) \( d = 0.756 \), where \( d \) is one plus the rate of capital loss on the stock if the price goes down.
(v) Calculate the true probability of the stock price going up.

The continuously compounded annual expected return on the stock is 10%.

(A) 0.52
(B) 0.57
(C) 0.62
(D) 0.67
(E) 0.72

**Solution:**

\[
p = \frac{e^\alpha - d}{u - d} = \frac{e^{0.1} - 0.756}{1.433 - 0.756} \approx 0.52.
\]

3.4. **Expected Rate of Return on the Stock under the Risk-neutral Probability.**

What if we calculate the expected wealth using \( p^* \) as the probability that the stock price goes up?

\[
p^*S_u + (1 - p^*)S_d = \frac{e^\tau - d}{u - d}S(0)u + \frac{u - e^\tau}{u - d}S(0)d = \frac{S(0)}{u - d}(ue^\tau - ud + ud - de^\tau) = e^\tau S(0)
\]

You probably expected the above result . . .
3.5. **Expected Rate of Return on an Option.** So, would it be appropriate to take the expected payoffs of the put options above and simply “discount” them using $\alpha$? This is not going to work since an investment in the option is not the same thing as an investment in the stock itself (if it were, there would be no need to invent options in the first place!). Remember, for instance, that the put options are rendered worthless in some nodes of the tree, while the stock is not.

Consider an option with the payoff $V_u$ in the “up” node and $V_d$ in the “down” node, and denote the initial price of this option by $V_p(0)$. The subscript $p$ emphasises the use of the “true” probability $p$. We introduce the “mean rate of return” $\gamma$ of the option. Then,

$$V_p(0)e^{\gamma} = pV_u + (1 - p)V_d \ \Rightarrow \ \ V_p(0) = e^{-\gamma}[pV_u + (1 - p)V_d].$$

(3.1)

So, now we get to have new notation, but this does not help us find the price of the option – just to connect the option price to its mean rate of return.