Notes: This is a closed book and closed notes exam.
Time: 50 minutes

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FOR GRADER’S USE ONLY:

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1.1. **TRUE/FALSE QUESTIONS.** *Please note your answers on the front page.*

**Problem 1.1.** When a caplet is paid in arrears, its price is strictly greater than when it is paid in advance. *True or false?*

**Solution:** FALSE

**Problem 1.2.** The price of a cap can be calculated as the sum of the prices of caplets it consists of. *True or false?*

**Solution:** TRUE

**Problem 1.3.** In our usual notation, in a Black-Derman-Toy tree, we have that

\[ r_{uu} > r_{ud} > r_{dd}. \]

*True or false?*

**Solution:** TRUE
1.2. **FREE-RESPONSE PROBLEMS.** Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they’re correct) are worth 0 points.

**Problem 1.4.** (10 points) The evolution of effective annual interest rates is modeled by a three-period tree so that during each period the interest rate can either increase by 0.02 or decrease by 0.01. Let the initial (root) interest rate be equal to 0.08. The risk-neutral probability of an up movement is 0.6 at every node.

Consider a three-year interest-rate cap with a cap rate equal to 0.09 purchased to hedge against adverse movements of the above-modeled floating rate. Let the loan amount be equal to $1000 and let the loan repayment installments be interest-only.

What is the price of the above cap consistent with our model?

**Solution:** The nodes in the tree at which the cap makes a positive payment, along with the respective payment amounts are:

- \( \text{up} : 1000(0.10 - 0.09) = 1000(0.01), \)
- \( \text{up} - \text{up} : 1000(0.12 - 0.09) = 1000(0.03). \)

So, the cap’s price equals

\[
1000 \times \frac{1}{1.08} \left[ 0.6 \times \frac{0.01}{1.1} + (0.6)^2 \times \frac{0.03}{1.12(1.1)} \right] = \frac{1000(0.6)(0.01)}{1.08(1.1)} \left[ 1 + \frac{0.6(3)}{1.12} \right] = \frac{6}{1.08(1.1)}(2.607143) = 13.16739
\]
Problem 1.5. (20 points) A three-period Black-Derman-Toy tree is calibrated so that
\[ r_0 = 0.03, \quad r_u = 0.04, \quad r_{uu} = 0.05, \]
\[ r_d = 0.035, \quad r_{ud} = 0.04. \]

Consider a one-year, $0.93$-strike put option on a zero-coupon bond which matures at time $3$ for $1$. What is the price of this put option consistent with the above interest-rate model?

Solution: The bond prices at the up and down nodes are
\[ P_u = \frac{1}{1.04} \times \frac{1}{2} \left[ \frac{1}{1.05} + \frac{1}{1.04} \right] = 0.92015. \]
\[ P_d = \frac{1}{1.035} \times \frac{1}{2} \left[ \frac{1}{1.032} + \frac{1}{1.04} \right] = 0.9326. \]

So, the price of the put is
\[ V_P(0) = \frac{1}{1.03} \times \frac{1}{2} (0.93 - 0.92015) = 0.00478155. \]
Problem 1.6. (10 points) The evolution of short-term interest rates is modeled using the Black-Derman-toy tree with base rate parameters

\[ r_0 = 0.06, \quad r_d = 0.05, \quad r_{dd} = 0.045, \quad r_{ddd} = 0.04, \]

and volatilities of effective interest rates

\[ \sigma_1 = 0.10, \quad \sigma_2 = 0.12, \quad \sigma_3 = 0.15. \]

Consider a four-year caplet with cap rate of 0.08 on a $1000 interest-only loan with end-of-year payment installments. What is today’s price of this caplet?

Solution: The interest rates relevant to the payoff of the caplet are

\[ r_{ddu} = 0.04e^{2(0.15)} = 0.05399, \quad r_{duu} = 0.04e^{4(0.15)} = 0.0729, \quad r_{uuu} = 0.04e^{6(0.15)} = 0.0984. \]

The payoffs is, therefore, only

\[ V_{uuu} = 1000(0.0984 - 0.08) = 1000(0.0184) = 18.40. \]

The other relevant interest rates are

\[ r_{uu} = 0.045e^{4(0.12)} = 0.0727, \quad r_u = 0.0611. \]

The price of this caplet is

\[ 18.40 \times \frac{1}{8} \times (1.0984)^{-1} \times (1.0727)^{-1} \times (1.0611)^{-1} \times (1.06)^{-1} = 1.7355. \]
Problem 1.7. (20 points) Here is an (incomplete) Black-Derman-Toy tree

\[ r_u = 0.04, \quad r_{uu} = 0.06, \]
\[ r_d = 0.03, \quad r_{ud} = 0.04. \]

Compute the volatility in year-1 of the 3-year zero-coupon bond generated by the above tree.

Solution: To complete the BDT tree, we have

\[ r_{dd} = \frac{0.04^2}{0.06} = 0.0267. \]

The two possible time−1 bond prices are

\[ P_u = \frac{1}{1.04} \times \frac{1}{2} \times \left[ \frac{1}{1.06} + \frac{1}{1.04} \right] = 0.915834, \]
\[ P_u = \frac{1}{1.03} \times \frac{1}{2} \times \left[ \frac{1}{1.0267} + \frac{1}{1.04} \right] = 0.939579. \]

The two yields are

\[ y_u = \frac{1}{\sqrt{P_u}} - 1 = 0.044941, \]
\[ y_d = \frac{1}{\sqrt{P_d}} - 1 = 0.031652 \]

The required volatility is, hence,

\[ \kappa = \frac{1}{2} \ln \left( \frac{0.044941}{0.031652} \right) = 0.1752746 \]
Problem 1.8. (20 points) A discrete-time model is used to model both the price of a non-dividend-paying stock and the short-term (risk-free) interest rate. Each period is one year.

The time−0 stock price is given to be $80 per share, while the one-year spot rate for zero-coupon bonds equals 0.04.

At time−1, there are two states of the world: sunny and cloudy. In the sunny state of the world, the stock price is $100 and the effective yearly interest rate is 0.03. In the cloudy state of the world, the stock price is $75 and the effective yearly interest rate is 0.05.

Consider a European put and a European call on the above stock whose strike is $90 and whose exercise date is at time−2. What is the difference between the two option prices?

Solution: The difference between the two options’ prices is

\[ V_C(0) - V_P(0) = F_{0.2}^P(S) - K P(0, 2) = S(0) - K P(0, 2) \]

The risk-neutral probability of the sunny state of the world is

\[ p^* = \frac{80(1.04) - 75}{100 - 75} = 0.328. \]

So,

\[ P(0, 2) = \frac{1}{1.04} \left[ \frac{1}{1.03} \times 0.328 + \frac{1}{1.05} \times (1 - 0.328) \right] = 0.9216. \]

Finally, our answer is

\[ 80 - 90(0.9216) = -2.9425. \]
1.3. **MULTIPLE CHOICE QUESTIONS.** Please, record your answers on the front page of this exam.

**Problem 1.9.** (5 points) The two-period interest-rate tree models effective annual interest rates. The two possible interest rates for the time period $[1, 2]$ are given to be 0.04 and 0.02. The risk-neutral probability of an up move is specified as 0.55.

The price of a zero-coupon bond redeemable in two years for $1 is $0.95.

What is the root effective interest rate consistent with the above bond prices?

(a) About 0.021.
(b) About 0.023.
(c) About 0.024.
(d) About 0.025.
(e) None of the above.

**Solution:** (a)

\[
0.95 = P(0, 1) \left[ \frac{0.55}{1.04} + \frac{0.45}{1.02} \right] \Rightarrow r_0 = 0.021.
\]

**Problem 1.10.** (5 points) The spot rates for zero-coupon bonds are observed to be

\[
r_0(0, 1) = 0.03, \quad r_0(0, 2) = 0.04, \quad r_0(0, 3) = 0.045, \quad r_0(0, 4) = 0.0475.
\]

What is the forward price for delivery in one year of a zero-coupon bond with three years left to maturity? Assume the redemption amount of $100.

(a) About $85.55
(b) About $90.26
(c) About $98.33
(d) About $98.56
(e) None of the above.

**Solution:** (a)

\[
F_{0,1}[P(1, 4)] = \frac{P(0, 4)}{P(0, 1)} = \frac{1 + r_0(0, 1)}{(1 + r_0(0, 4))^4} = 0.8555.
\]

**Problem 1.11.** (5 points) It is observed that the bond prices for zero coupon bonds redeemable for $1 are, in our usual notation,

\[
P(0, 1) = 0.9615 \quad \text{and} \quad P(0, 2) = 0.9157.
\]

In the Black-Derman-Toy tree, the base rate parameter is set to equal $r_d = 0.035$. What is the interval into which the value of the volatility of effective interest rates in year-2 falls?

(a) $[0, 0.1)$
(b) $[0.1, 0.15)$
(c) $[0.15, 0.25)$
(d) $[0.25, 0.35)$
(e) None of the above.

**Solution: (d)**

We need to solve for the parameter $\sigma_1$ in the calibration equation. The calibration equation is

$$P(0, 2) = P(0, 1) \times \frac{1}{2} \left[ \frac{1}{1 + r_d} + \frac{1}{1 + r_d e^{2\sigma_1}} \right].$$

In the present problem, we have

$$0.9157 = 0.9615 \times \frac{1}{2} \left[ \frac{1}{1.035} + \frac{1}{1 + 0.035 e^{2\sigma_1}} \right].$$

So,

$$1.9047 = \frac{1}{1.035} + \frac{1}{1 + 0.035 e^{2\sigma_1}} \quad \Rightarrow \quad 1 + 0.035 e^{2\sigma_1} = 1.0168 \quad \Rightarrow \quad \sigma_1 = 0.3134.$$