Name:

M339W/389W Financial Mathematics for Actuarial Applications
University of Texas at Austin
The Prerequisite In-Term Exam
Instructor: Milica Ćudina

Notes: This is a closed book and closed notes exam. The maximum number of points on this exam is 100.
Time: 50 minutes

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FOR GRADER’S USE ONLY:

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1.1. **DEFINITION.**

**Problem 1.1.** (10 points)
Provide the definition of an **arbitrage portfolio**.

**Solution:** See your lecture notes from *M339D*.
1.2. True/false questions.

**Problem 1.2.** (2 pts) If \(X\) and \(Y\) are independent random variables, then
\[
F_{X+Y}(a) = F_X(a) \cdot F_Y(a).
\]
True or false?

**Solution:** FALSE

**Problem 1.3.** (2 pts) If the random variable \(X\) is standard normal, then the distribution function of the random variable \(Y = |X|\) equals
\[
F_Y(a) = 2\Phi(a) - 1 \text{ for every } a \geq 0.
\]
True or false?

**Solution:** TRUE

For every \(a \geq 0,
\begin{align*}
F_Y(a) &= \Pr[Y \leq a] \\
&= \Pr[|X| \leq a] \\
&= \Pr[-a \leq X \leq a] \\
&= \Pr[X \leq a] - \Pr[X < -a] \\
&= \Pr[X \leq a] - (1 - \Pr[X \geq -a]) \\
&= 2\Phi(a) - 1.
\end{align*}

**Problem 1.4.** (2 pt) Let \(X\) be a normal random variable with parameters \((\mu = 2, \sigma^2 = 1)\), and let \(Y\) be a normal random variable with parameters \((\mu = -2, \sigma^2 = 1)\). Assume that \(X\) and \(Y\) are independent. Then, the variance of the random variable \(X + Y\) equals 2. True or false?

**Solution:** TRUE

See the “Addition rule for variances”.

**Problem 1.5.** (2 points) In our usual notation, let \(S(0) = 40, r = 0.08, \sigma = 0.3, \delta = 0\). You need to construct a 2–period forward binomial tree for the above stock with every period in the tree of length \(h = 0.5\). Then, \(u > 1.45\). True or false?

**Solution:** FALSE

\[
u = \exp\{(0.08 - 0) \cdot 0.5 + 0.3\sqrt{0.5}\} \approx 1.29.
\]

**Problem 1.6.** (2 points) It is never optimal to exercise an American call option on a non-dividend paying stock early. True or false?
Solution: TRUE

Problem 1.7. (2 points) You are using a binomial asset-pricing model to model the evolution of the price of a particular stock. Then, the Δ in the replicating portfolio of a single call option on that stock never exceeds 1. True or false?

Solution: TRUE
The call’s Δ will always be between 0 and 1.

1.3. Free-response problems. Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they’re correct) are worth 0 points.

Problem 1.8. (6 points) Let $X \sim N(0, 1)$. Find:

(i) $P[-1.79 \leq X \leq -0.54]$

(ii) $P[X \geq 1.13]$

(iii) $P[|X| \leq 0.5]$

Solution: Using the tables for the standard normal distribution function, we get

(i) 0.2579
(ii) 0.1292
(iii) 0.3830
Problem 1.9. (15 points) Source: Sample P Exam problem #18.

Two instruments are used to measure the height $h$ of a tower. The error made by the less accurate instrument is normally distributed with mean 0 and standard deviation 0.0056$h$. The error made by the more accurate instrument is normally distributed with mean 0 and standard deviation 0.0044$h$.

The errors from the two instruments are independent of each other. Calculate the probability that the average value of the two measurements is within 0.005$h$ of the height of the tower.

Solution: Let $X_1$ and $X_2$ denote the measurement errors of the less and more accurate instruments, respectively. Then,

$$X_1 \sim N(mean = 0, var = (0.0056h)^2), \quad X_2 \sim N(mean = 0, var = (0.0044h)^2).$$

Since $X_1$ and $X_2$ are also independent, their average has the distribution

$$\bar{X} = \frac{1}{2}(X_1 + X_2) \sim N(mean = 0, var = (0.00356h)^2).$$

So, rewriting $\bar{X}$ in standard units, we get

$$P[|\bar{X}| < 0.005h] = P \left[ \left| \frac{\bar{X} - 0}{0.00356h} \right| < 1.4 \right] = 2\Phi(1.4) - 1 = 0.8384.$$
Problem 1.10. (5 points) An investor short sells one share of a non-dividend-paying stock and writes an at-the-money, $T$-year, European put option on this stock. The put premium is denoted by $V_P(0)$. Assume that there are no transaction costs. The continuously compounded, risk-free interest rate is denoted by $r$. Let the argument $s$ represent the stock price at time $T$.

(i) (3 points) Determine an algebraic expression for the investor’s profit at expiration $T$ in terms of $V_P(0), r, T$ and the strike $K$.

(ii) (2 points) In particular, how does the expression you obtained in (i) simplify if the put is in-the-money on the exercise date?

Solution:

$$-s - (K - s) + (S(0) + V_P(0))e^{rT} = -s - (K - s) + (K + V_P(0))e^{rT}. $$

For the option to be in-the-money at expiration, we must have $s < K$. So, the profit simplifies to

$$-s - (K - s) + (K + V_P(0))e^{rT} = -K + (K + V_P(0))e^{rT}. $$
Problem 1.11. (5 points)
Let $X$ be a continuous random variable with probability density function $f_X(x)$. Let its cumulative distribution function be denoted by $F_X(x) = \mathbb{P}[X \leq x]$. Define the new random variable $Y$ as

$$Y = F_X(X).$$

Find $\mathbb{E}[Y]$.

Solution: The connection between the probability density function and the cumulative distribution function is

$$f_X(x) = F_X'(x).$$

Using the definition of the expected value of a function of a continuous random variable, we get

$$\mathbb{E}[Y] = \mathbb{E}[F_X(X)] = \int_{-\infty}^{\infty} F_X(x) f_X(x) \, dx.$$ 

We can use a change of variables $u = F_X(x)$ with which $du = f_X(x) \, dx$. So, our result is

$$\mathbb{E}[Y] = \int_0^1 u \, du = 1/2.$$
**Problem 1.12.** (10 pts) For a two-period binomial model, you are given that:

1. each period is one year;
2. the current price of a non-dividend-paying stock $S$ is $S(0) = $20;
3. $u = 1.2$, with $u$ as in the standard notation for the binomial model;
4. $d = 0.8$, with $d$ as in the standard notation for the binomial model;
5. the continuously compounded risk-free interest rate is $r = 0.04$.

Consider a **special** call option which pays the excess above the strike price $K = 23$ (if any!) at the end of **every** binomial period.

Find the **price** of this option.

**Solution:** The risk-neutral probability is

$$p^* = \frac{e^{0.04} - 0.8}{1.2 - 0.8} = 0.6020.$$

When one constructs the two-period binomial tree, one gets

$$S_u = 24, S_d = 16,$$

$$S_{uu} = 28.80, S_{ud} = S_{dd} = 19.2, S_{dd} = 12.8.$$

So, the payoffs at the end of the first period are

$$V_u = 1, V_d = 0.$$

The payoffs at the end of the second period are

$$V_{uu} = 5.80, V_{ud} = 0, V_{dd} = 0.$$

So, taking the expected value at time 0 of the payoff with respect to the risk-neutral probability, we get that the price of this call should be

$$e^{-0.04} \times V_u \times p^* + e^{-0.04 \times 2}[V_{uu} \times (p^*)^2 + V_{ud} \times 2p^*(1 - p^*)]$$

$$= e^{-0.04} \times 1 \times 0.6020 + e^{-0.08}[5.8 \times 0.6020^2]$$

$$= 2.51893.$$
1.4. **MULTIPLE CHOICE QUESTIONS.** Please note your answers on the front page.

**Problem 1.13.** (5 pts) Let \( X \) be a normally distributed random variable with mean \( \mu = 2 \) and standard deviation equal to \( \sigma = 1 \). Find \( \mathbb{E}[X^2] \).

(a) 2  
(b) 3  
(c) 4  
(d) 5  
(e) None of the above

**Solution:** (d)  
\[ \mathbb{E}[X^2] = \text{Var}[X] + \mathbb{E}[X]^2 = 5. \]

**Problem 1.14.** (5 pts) Assume that the price of a two-year zero-coupon bond is 0.88 and that the price of a three-year zero-coupon bond is 0.85. The two bonds have the same redemption amounts of $1. Denote by \( f \) the implied forward interest rate from year 2 to year 3. Then:

(a) \( 0 \leq f \leq 0.01 \)  
(b) \( 0.01 \leq f \leq 0.05 \)  
(c) \( 0.05 \leq f \leq 0.09 \)  
(d) \( 0.09 \leq f \leq 0.015 \)  
(e) None of the above

**Solution:** (b)  
Let \( P(0, 2) \) denote the price of a two-year zero-coupon bond and let \( P(0, 3) \) denote the price of a three-year zero-coupon bond. Then,  
\[ f_{[2,3]} = \frac{(1 + r_3)^3}{(1 + r_2)^2} - 1 = \frac{P(0, 2)}{P(0, 3)} - 1 = \frac{0.88}{0.85} - 1 \approx 0.0352. \]

**Problem 1.15.** (5 pts) *Source: Problem 16a (p.331) from Kellison.*  
You are given the following table of spot rates:

<table>
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<th>Length of Investment</th>
<th>Spot rate</th>
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<tr>
<td>2 years</td>
<td>0.0800</td>
</tr>
<tr>
<td>3 years</td>
<td>0.0875</td>
</tr>
<tr>
<td>4 years</td>
<td>0.0925</td>
</tr>
<tr>
<td>5 years</td>
<td>0.0950</td>
</tr>
</tbody>
</table>

Find the price of a $1,000 three-year bond with annual 5% coupons.

(a) About 800
(b) About 850
(c) About 900
(d) About 950
(e) None of the above

Solution: (c)

\[50 \cdot (1 + 0.07)^{-1} + 50 \cdot (1 + 0.08)^{-2} + 1050 \cdot (1 + 0.0875)^{-3} \approx 905.99.\]
**Problem 1.16.** (5 pts) *Source: Sample FM Problem #26.*

A 5-year loan for 10,000 is charged a nominal interest rate of 12% compounded semianually. The loan is to be repaid so that interest is repaid at the end of every 6 month period as it accrues and the principal is repaid in total at the end of the 5 years.

Denote the total amount of interest paid on this loan by $I$. Then

(a) $I \approx 2,750$
(b) $I \approx 3,000$
(c) $I \approx 3,250$
(d) $I \approx 3,500$
(e) None of the above

**Solution:** (e)

$$10 \cdot \frac{0.12}{2} \cdot 10,000 = 6,000.$$ 

**Problem 1.17.** (5 points) Let $X$ and $Y$ be independent and normally distributed. Assume that $X$ is normal with mean 0 and standard deviation 3 and that $Y$ has mean 1 and standard deviation 5. Find $P[X - Y > 0]$.

(a) 0.5793
(b) 0.5160
(c) 0.5080
(d) 0.5
(e) None of the above

**Solution:** (e)

The distribution of $X - Y$ is normal with mean $-1$. So,

$$P[X - Y > 0] < P[X - Y > -1] = 0.5.$$ 

**Problem 1.18.** Which of the following formulas hold for the exponential function:

(a) $e^x + e^y = e^{x+y}$
(b) $e^x e^y = e^x + e^y$
(c) $e^{x+y} = e^x e^y$
(d) $e^{x-y} = e^x - e^y$
(e) None of the above.

**Solution:** The correct answer is (c).
Problem 1.19. (5 pts)
Harry takes a $1,000, 2-year loan from Roger. Assume that he repays the loan with end-of-year, interest-only payments and a final payment of $1,000 at the end of year 2. Neither Roger, nor Harry is particularly risk-averse, so they decide to define the interest charged in the following way:

- At time$-0$, they toss a fair coin. If the result of the coin toss is Heads, the effective interest rate charged for the first year is 0.03. If the result of the coin toss is Tails, the effective interest rate charged for the first year is 0.04.
- At time$-1$, they roll a fair die. If the result is divisible by three, the effective interest rate charged for the second year is 0.04. Otherwise, the effective interest rate charged for the second year is 0.01.

What is the expected amount of interest Harry pays according to the above scheme?
(a) 45
(b) 50
(c) 55
(d) 60
(e) None of the above

Solution: (c)
The expected amount of interest paid at time$-1$ is
\[1000 \times \frac{1}{2}(0.03 + 0.04) = 35\]
The expected amount of interest paid at time$-2$ is
\[1000 \times \left(\frac{1}{3}(0.04) + \frac{2}{3}(0.01)\right) = 20.\]
So, the total expected amount of interest equals 55.

Problem 1.20. A coin is tossed, and, independently, a 6-sided die is rolled. Let
\[A = \{4 \text{ is obtained on the die}\}\]
\[B = \{\text{Heads is obtained on the coin and an even number is obtained on the die}\}\].

Then
(a) \(A \) and \(B\) are mutually exclusive
(b) \(A \) and \(B\) are independent
(c) \(A \subseteq B\)
(d) \(A \cap B = B\)
(e) None of the above.

Solution: The correct answer is (e).