Notes: This is a closed book and closed notes exam. The maximum number of points on this exam is 100.

Time: 50 minutes

MULTIPLE CHOICE

| 1.9 (5) | a | b | c | d | e |
| 1.10 (5) | a | b | c | d | e |
| 1.11 (5) | a | b | c | d | e |
| 1.12 (5) | a | b | c | d | e |
| 1.13 (5) | a | b | c | d | e |

FOR GRADER’S USE ONLY:

<table>
<thead>
<tr>
<th>DEF</th>
<th>T/F</th>
<th>1.2</th>
<th>1.3</th>
<th>1.4</th>
<th>1.5</th>
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<th>1.7</th>
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<th>M.C.</th>
<th>Σ</th>
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1.1. **DEFINITION.**

**Problem 1.1.** (10 points)  
Provide the definition of an **arbitrage portfolio**.

**Solution:** See the lecture notes.
1.2. Free-response problems. Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they’re correct) are worth 0 points.

Problem 1.2. (15 points)
Two scales are used to measure the mass \( m \) of a precious stone. The first scale makes an error in measurement which we model by a normally distributed random variable with mean \( \mu_1 = 0 \) and standard deviation \( \sigma_1 = 0.04m \). The second scale is more accurate. We model its error by a normal random variable with mean \( \mu_2 = 0 \) and standard deviation \( \sigma_2 = 0.03m \).

We assume that the measurements made using the two different scales are independent.
To get our final estimate of the mass of the stone, we take the average of the two results from the two different scales.
What is the probability that the value we get is within 0.005\( m \) of the actual mass of the stone?

Solution: Let us denote the random variable modeling the error from the first scale by \( X_1 \sim N(0, \sigma_1^2) \) and the random variable modeling the error from the second scale by \( X_2 \sim N(0, \sigma_2^2) \).
Then, if \( Y \) denotes the average of the two measurements, we have that
\[
Y = \frac{1}{2} (X_1 + X_2) \sim N(0, \frac{1}{4}(\sigma_1^2 + \sigma_2^2)),
\]
i.e.,
\[
Y \sim N(0, \sigma^2)
\]
with
\[
\sigma^2 = \frac{1}{4} (\sigma_1^2 + \sigma_2^2) = \frac{1}{4} (0.04^2 m^2 + 0.03^2 m^2) = \frac{1}{4} \cdot 0.01^2 m^2 (4^2 + 3^2) = \frac{1}{4} \cdot 0.05^2 m^2 = \left( \frac{0.05m}{2} \right)^2.
\]
The probability we are looking for can be expressed as
\[
P[Y \in (-0.005m, 0.005m)] = P[-0.005m < Y < 0.005m] = P[-2 \cdot \frac{0.005m}{0.05m} < \frac{Y}{\sigma} < 2 \cdot \frac{0.005m}{0.05m}] = P[-0.2 < \frac{Y}{\sigma} < 0.2].
\]
Since \( \frac{Y}{\sigma} \sim N(0, 1) \), the above probability equals
\[
2\Phi(0.2) - 1 \approx 2 \cdot 0.57926 - 1 \approx 0.16.
\]
Problem 1.3. (10 points) Assume that $Y_1 = e^X$ where $X$ is a standard normal random variable.

(i) (2 points) What is the probability that $Y_1$ exceeds 5?

(ii) (3 + 5 points) Find the mean and the variance of $Y_1$.

*Hint:* It helps if you use the expression for the moment generating function of a standard normal random variable.

Solution:

(i) 

\[ P[Y_1 > 5] = P[e^X > 5] = P[X > \ln(5)] = 1 - N(\ln(5)) \approx 1 - N(1.61) = 1 - 0.9463 = 0.0537. \]

(ii)

\[
\mathbb{E}[Y_1] = \mathbb{E}[e^X] = \mathbb{E}[e^{1 \cdot X}] = M_X(1)
\]

where $M_X$ denotes the moment generating function of $X$. In class, we recalled the following expression for $M_X$:

\[ M_X(t) = e^{t^2/2}. \]

So, $\mathbb{E}[Y_1] = e^{1/2} = \sqrt{e}$.

The second moment of $Y_1$ is obtained similarly as

\[ \mathbb{E}[Y_1^2] = \mathbb{E}[e^{2 \cdot X}] = M_X(2) = e^2. \]

So,

\[ \text{Var}[Y_1] = \mathbb{E}[Y_1^2] - (\mathbb{E}[Y_1])^2 = e^2 - e = e(e - 1). \]
Problem 1.4. (5 points) Source: FM(DM) sample problem #42.
An investor purchases one share of a non-dividend-paying stock and writes an at-the-money, $T$-year, European call option in this stock. The call premium is denoted by $C$. Assume that there are no transaction costs. The continuously compounded, risk-free interest rate is denoted by $r$. Let the argument $s$ represent the stock price at time $T$.

(i) (3 points) Determine an algebraic expression for the investor’s profit at expiration $T$ in terms of $C, r, T$ and the strike $K$.

(ii) (2 points) In particular, how does the expression you obtained in (i) simplify if the call is in-the-money on the exercise date?

Solution:

$$s - (s - K)_{+} - (S(0) - C)e^{rT} = s - (s - K)_{+} - (K - C)e^{rT}.$$ 

For $s > K$,

$$s - (s - K)_{+} - (K - C)e^{rT} = K(1 - e^{rT}) + Ce^{rT}.$$
Problem 1.5. (5 points) Harry graduated with an art degree 10 years ago. He subsequently became an ASA. Which one of the following statements do you think is more likely to apply to Harry:

a. Harry now works at a consultancy.

b. Harry now works at a consultancy and enjoys volunteering at the Museum of Modern Art on the weekends.

Substantiate your choice with a short explanation.

Solution: a. is more likely since b. is a subset of a.

Problem 1.6. (15 points) The random vector \((X_1, X_2, X_3)\) is jointly normal. Its marginal distributions are:

\[
X_1 \sim N(\text{mean} = 0, \text{variance} = 4), \quad X_2 \sim N(\text{mean} = 1, \text{variance} = 1), \quad X_3 \sim N(\text{mean} = -1, \text{variance} = 9).
\]

The correlation coefficients are given to be

\[
corr[X_1, X_2] = 0.3, \quad corr[X_2, X_3] = 0.4, \quad corr[X_1, X_3] = -0.3.
\]

What is the distribution of the random variable \(X = X_1 - X_2 + 2X_3\)? Please, provide the name of the distribution, as well as the values of its parameters.

Solution:

(2 points) The linear combination of jointly normal random variables in normally distributed itself. Now, we need to identify the mean and the variance of this normal distribution.

(3 points) The mean of \(X\) is

\[
E[X] = E[X_1] - E[X_2] + 2E[X_3] = 0 - 1 + 2(-1) = -3.
\]

(10 points) The variance of \(X\) is

\[
Var[X] = Var[X_1] + Var[X_2] + 4Var[X_3] - 2Cov[X_1, X_2] + 4Cov[X_1, X_3] - 4Cov[X_2, X_3]
\]

\[
= 4 + 1 + 36 - 2(2)(1)(0.3) + 4(2)(3)(-0.3) - 4(1)(3)(0.4) = 27.8.
\]
Problem 1.7. (10 pts) For a two-period binomial model, you are given that:

1. each period is one year;
2. the current price of a non-dividend-paying stock \( S \) is \( S(0) = $20 \);
3. \( u = 1.3 \), with \( u \) as in the standard notation for the binomial model;
4. \( d = 0.9 \), with \( d \) as in the standard notation for the binomial model;
5. the continuously compounded risk-free interest rate is \( r = 0.05 \).

Consider a special call option which pays the excess above the strike price \( K = 23 \) (if any!) at the end of every binomial period.

Find the price of this option.

Solution: The risk-neutral probability is
\[
p^* = \frac{e^{0.05} - 0.9}{1.3 - 0.9} = 0.3782.
\]

When one constructs the two-period binomial tree, one gets
\[
S_u = 26, \quad S_d = 17,
\]
\[
S_{uu} = 33.80, \quad S_{ud} = S_{dd} = 23.4, \quad S_{dd} = 16.2.
\]

So, the payoffs at the end of the first period are
\[
V_u = 3, \quad V_d = 0.
\]

The payoffs at the end of the second period are
\[
V_{uu} = 10.80, \quad V_{ud} = 0.4, \quad V_{dd} = 0.
\]

So, taking the expected value at time 0 of the payoff with respect to the risk-neutral probability, we get that the price of this call should be
\[
e^{-0.05} \times V_u \times p^* + e^{-0.05 \times 2}[V_{uu} \times (p^*)^2 + V_{ud} \times 2p^*(1 - p^*)]
= e^{-0.05} \times 3 \times 0.3782 + e^{-0.1}[10.8 \times 0.3782^2 + 0.4 \times 2 \times 0.3782 \times (1 - 0.3782)]
= 1.079 + 1.568 = 2.647.
\]
Problem 1.8. (5 points)

Emmanuel entered an extra special kind of game with his friend Fischer. First, they toss a fair coin. If the coin comes up heads, Emmanuel gives $5,000 to Fischer. If the coin comes up tails, Fischer gives $2,000 to Emmanuel. Then, regardless of the outcome of the first cointoss, they toss the same fair coin again. If it comes up heads, Emmanuel gives Fischer $4,000. If the coin comes up tails, Fischer gives $3,000 to Emmanuel. What is the expected cashflow, i.e., what is the expected amount of money that changes hands and who gives it to whom?

Solution: Let $X_i$ be the cashflow from Emmanuel’s perspective after the $i^{th}$ cointoss for $i = 1, 2$. Then, we are looking for

$$E[X_1 + X_2] = E[X_1] + E[X_2]$$

$$= \frac{1}{2}(-5,000 + 2,000) + \frac{1}{2}(-4,000 + 3,000) = -2,000.$$

So, the expected cashflow is $2,000 from Emmanuel to Fischer.

1.3. MULTIPLE CHOICE QUESTIONS. Please note your answers on the front page.

Problem 1.9. (5 pts) Consider a continuous-dividend-paying stock currently priced at $100 per share whose dividend yield is projected to equal 0.02.

The price of this stock in one year is modeled using a one-period binomial tree. The down factor $d$ is given to be 0.83, while the up factor $u$ is unknown. (If you want to you can construct a cutesy narrative about coffee being spilled on the model write-up or some such.)

The continuously compounded risk-free interest rate is given to equal 0.04.

You observe that the price of a one-year, $110-strike European call option on this stock consistent with the above model equals $5.15. Find the price of the otherwise identical put option.

(a) $10.84
(b) $12.82
(c) $15.15
(d) $17.13
(e) None of the above.

Solution: (b)

By put-call parity

$$V_P(0) = V_C(0) + Ke^{-rT} - S(0)e^{-\delta T} = 5.15 + 110e^{-0.04} - 100e^{-0.02} \approx 12.82.$$
Problem 1.10. (5 pts) Consider a non-dividend-paying stock currently priced at $100 per share. The price of this stock in one year is modeled using a one-period binomial tree under the assumption that the stock price can either go up to 110 or down to 90.

Let the continuously compounded risk-free interest rate equal 0.04. What is the risk-neutral probability of the stock price going up?

(a) About 0.2969
(b) About 0.3039
(c) About 0.5000
(d) About 0.7041
(e) None of the above.

Solution: (d)

\[ p^* = \frac{100e^{0.04} - 90}{110 - 90} = 0.7041. \]

Problem 1.11. (5 pt) Let a yield curve be defined by the following equation:

\[ i_k = 0.09 + 0.002k - 0.001k^2 \]

where \( i_k \) denotes the annual effective rate of return for zero coupon bonds with maturity of \( k \) years. Let \( r_0(4,5) \) be the one-year effective forward rate during year 5 that is implied by this yield curve.

Calculate \( f = r_0(4,5) \).

(a) About 3.6%
(b) About 4.7%
(c) About 5.6%
(d) About 6.8%
(e) None of the above

Solution: (b)

Let \( f = r_0(4,5) \). By definition, \( f \) should satisfy the equation

\[ 1 + f = \frac{(1 + i_5)^5}{(1 + i_4)^4}. \]

From the equation for the yield curve given in the problem, we get that \( i_4 = 0.082 \) and \( i_5 = 0.075 \). Plugging these numbers into the above equation and solving for \( f \), we obtain that \( f = 0.047 \).

Problem 1.12. (5 pts) Assume that the price of a two-year zero-coupon bond is 0.881659 and that the price of a three-year zero-coupon bond is 0.816298. The two bonds have the same redemption amounts of $1.

Denote by \( f = r_0(2,3) \) the implied forward interest rate from year 2 to year 3. Then:

(a) \( 0 \leq f \leq 0.01 \)
(b) \( 0.01 \leq f \leq 0.05 \)
(c) \( 0.05 \leq f \leq 0.09 \)
(d) $0.09 \leq f \leq 0.015$
(e) None of the above

**Solution:** (c)
Let $P(0, 2)$ denote the price of a two-year zero-coupon bond and let $P(0, 3)$ denote the price of a two-year zero-coupon bond. Then,

$$f = \frac{(1 + r_3)^3}{(1 + r_2)^2} - 1 = \frac{P(0, 2)}{P(0, 3)} - 1 = \frac{0.881659}{0.816298} - 1 = 0.08.$$

**Problem 1.13.** (5 pts) You are given the following table of spot rates:

<table>
<thead>
<tr>
<th>Length of Investment</th>
<th>Spot rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 year</td>
<td>0.041</td>
</tr>
<tr>
<td>2 years</td>
<td>0.045</td>
</tr>
<tr>
<td>3 years</td>
<td>0.051</td>
</tr>
</tbody>
</table>

A 2 year par-value bond with annual coupons and the unknown coupon rate $r$ is sold at par. Find $r$.

(a) About 4.5%
(b) About 5.6%
(c) About 5.7%
(d) About 6.3%
(e) None of the above

**Solution:** (a)
Let $P$ denote the price (and the par value and the redemption amount). Then

$$P = Pr(1 + i_1)^{-1} + P(1 + r)(1 + i_2)^{-2}.$$

So,

$$r = \frac{1 - (1 + i_2)^{-2}}{(1 + i_1)^{-1} + (1 + i_2)^{-2}} = \frac{1 - (1.045)^{-2}}{(1.041)^{-1} + (1.045)^{-2}} \approx 0.0449.$$