2.1. Warm up.

**Problem 2.1.** Consider a non-dividend-paying stock currently priced at $100 per share.

The price of this stock in one year is modeled using a one-period binomial tree under the assumption that the stock price can either go up to 110 or down to 90.

- Let the continuously compounded risk-free interest rate equal 0.04.
- What is the risk-neutral probability of the stock price going up?
- What is the price of a one-year, at-the-money call option on the above stock?

**Solution:**

\[
p^* = \frac{100e^{0.04} - 90}{110 - 90} = 0.7041.
\]

\[
V_C(0) = e^{-0.04} \times 10 \times 0.7041 = 6.7649.
\]

2.2. “True” probabilities.

**Problem 2.2.** MFE Spring 2007: Problem #2

For a one-period binomial model for the price of a stock, you are given:

(i) The period is one year.

(ii) The stock pays no dividends.

(iii) \( u = 1.433 \), where \( u \) is one plus the rate of capital gain on the stock if the price goes up.

(iv) \( d = 0.756 \), where \( d \) is one plus the rate of capital loss on the stock if the price goes down.

(v) Calculate the true probability of the stock price going up.

The continuously compounded annual expected return on the stock is 10%.

(A) 0.52

(B) 0.57

(C) 0.62

(D) 0.67

(E) 0.72

**Solution:**

\[
p = \frac{e^{0.1} - d}{u - d} = \frac{1.433 - 0.756}{1.433 - 0.756} \approx 0.52.
\]

**Problem 2.3.** There are two possible states of the world in one year: the *bad* and the *good*. Our subjective opinion is that the probability of the *bad* state of the world is 1/3.

In our market-model there are two assets: one risky and one riskless. The risky asset will be worth $50 in the *bad* state of the world or $80 in the *good* state of the world. This asset pays no dividends and its continuously compounded rate of return consistent with our subjective probability equals 8%. The riskless asset has a single $100 payment in one year regardless of the state of the world. Its price today is $92.

Find the price today of a $70-strike, one-year European cash call option on the risky asset.

**Solution:** The price of a $70-strike, one-year European cash call option on the risky asset can be evaluated as:

\[
V_{CC}(0) = e^{-rT} \times \mathbb{P}^*[S(T) \geq K]
\]

where \( \mathbb{P}^* \) denotes the risk-neutral probability measure on the states of the world on the exercise date \( T \), \( K \) is the strike price and \( r \) is the continuously compounded risk-free interest rate.

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From the given prices of the risky asset in one year we conclude that
\[ \mathbb{P}^*[S(T) \geq 70] = p^* \]
where \( p^* \) denotes the risk-neutral probability of the good state of the world. By definition,
\[ p^* = \frac{e^r (S(0) - S_{bad})}{S_{good} - S_{bad}}. \]
So, we need the initial price of the risky asset \( S(0) \). The given continuously compounded rate of return \( \alpha \) is useful here since
\[ S(0) = e^{-\alpha T} \left[ S_{bad} \times p_{bad} + S_{good} \times p_{good} \right] = e^{-0.08} \left[ 50 \times \frac{1}{3} + 80 \times \frac{2}{3} \right] = 70e^{-0.08} = 64.62. \]
So,
\[ p^* = \frac{64.62 e^r - 50}{80 - 50} = \frac{64.62 e^r - 50}{30}. \]
Finally,
\[ V_{CC}(0) = e^{-r} \times \frac{64.62 e^r - 50}{30} = \frac{64.62 - 50e^{-r}}{30} = \frac{64.62 - 50 \times 0.92}{30} = 0.6207. \]

**Problem 2.4.** A non-dividend-paying stock is currently trading at $100 per share. The price of this stock in a year is modeled using a one-period forward binomial tree with the stock’s volatility specified as 0.30. The investor believes that the “true” probability of the stock price going up equals 0.46.

The continuously compounded risk-free interest rate equals 0.04.

- What is the continuously compounded rate of return on the above stock under the “true” probability the investor uses to model?
- Consider a $95-strike, one-year European call option on the above stock. What is the continuously compounded rate of return of this call option under the above “true” probability?
- What is the compounded rate of return of the otherwise identical put option under the above “true” probability?

**Solution:** The up and down factors in the forward tree are
\[ u = e^{(r-\delta)h + \sigma \sqrt{h}} = e^{0.04 + 0.3} = e^{0.34} = 1.4049, \]
\[ d = e^{(r-\delta)h - \sigma \sqrt{h}} = e^{0.04 - 0.3} = e^{-0.26} = 0.7711. \]
Therefore, the two possible stock-prices after the one binomial step equal \( S_u = 140.49 \) and \( S_d = 77.11 \).

- The continuously compounded rate of return on the above stock under the “true” probability the investor uses is the constant \( \alpha \) such that
\[ S(0)e^\alpha = p S_u + (1 - p) S_d. \]
We get
\[ \alpha = \ln \left( \frac{0.46 \times 140.49 + 0.54 \times 77.11}{100} \right) = 0.0608. \]
- The risk-neutral probability in this model is
\[ p^* = \frac{1}{1 + e^{\sigma \sqrt{h}}} = \frac{1}{1 + e^{0.3}} = 0.4256. \]
So, the call option’s price is
\[ V_C(0) = e^{-0.04} \times 0.4256 \times (140.49 - 95) = 18.60. \]

The call option’s continuously compounded rate of return under the above “true” probability \( p = 0.46 \) is the constant \( \gamma_C \) satisfying the following equation
\[ V_C(0)e^{\gamma_C} = 18.60e^{\gamma_C} = 0.46(140.49 - 95) = 20.9254. \]
So, \( \gamma_C = 0.1178. \)
The price of the put option is
\[ V_P(0) = e^{-0.04} \times (1 - 0.4256) \times (95 - 77.11) = 9.87. \]

The put option’s continuously compounded rate of return under the above “true” probability is the constant \( \gamma_P \) satisfying the equation
\[ V_P(0)e^{\gamma_P} = 9.87e^{\gamma_P} = 0.54(95 - 77.11). \]
So, \( \gamma_P = -0.0214 \).