Problem 2.1. BDT tree from specified volatilities

Let the current effective annual spot rates be

\[ r_0(0,1) = 0.04, \quad r_0(0,2) = 0.045, \quad r_0(0,3) = 0.05. \]

Additionally, we assume that the BDT tree is constructed under the assumption that the volatility of the annual effective one-year spot rates in one year is \( \sigma_1 = 0.08 \) and that the volatility of the annual effective one-year spot rates in two years is \( \sigma_2 = 0.10 \). Let us construct the complete BDT tree based on the above information.

Solution:

\[ r_{uu} = 0.0727 \]
\[ r_u = 0.0568 \]
\[ r_0 = 0.04 \quad r_{ud} = 0.0595 \]
\[ r_d = 0.0483 \]
\[ r_{uu} = 0.0487 \]

Problem 2.2. Forward on a bond

Consider the following values of interest rates from an incomplete Black-Derman-Toy interest rate tree for the effective annual interest rates.

\[ r_u = 0.35, \quad r_{uu} = 0.40, \]
\[ r_d = 0.25, \quad r_{dd} = 0.20. \]

Let \( F \) denote the forward price for delivery at time \(-2\) of a zero-coupon bond redeemable at time \(-3\) for $1000. Then,

(a) \( 0 \leq F < 757 \)
(b) \( 757 \leq F < 767 \)
(c) \( 767 \leq F < 857 \)
(d) \( 857 \leq F < 915 \)
(e) None of the above.

Solution: (c)

As we have seen in class, the above forward price can be obtained as

\[ F = 1000 \times \frac{P(0,3)}{P(0,2)} \]

where \( P(0,T) \) denotes the price of a zero-coupon bond redeemable at time \(-T\) for $1. Using the provided tree, we get

\[ P(0,2) = \frac{1}{1 + r_0} \times \frac{1}{2} \times \left[ \frac{1}{1 + r_u} + \frac{1}{1 + r_d} \right] \]

and

\[
\begin{align*}
P(0,3) &= \frac{1}{1 + r_0} \times \frac{1}{4} \times \left[ \frac{1}{1 + r_u} \times \left( \frac{1}{1 + r_{uu}} + \frac{1}{1 + r_{ud}} \right) + \frac{1}{1 + r_d} \times \left( \frac{1}{1 + r_{dd}} + \frac{1}{1 + r_{ud}} \right) \right] \\
&= \frac{1}{1 + r_0} \times \frac{1}{4} \times \left[ \frac{1}{1 + r_u} \times \left( \frac{1}{1 + r_{uu}} + \frac{1}{1 + \sqrt{r_{uu}r_{dd}}} \right) + \frac{1}{1 + r_d} \times \left( \frac{1}{1 + r_{dd}} + \frac{1}{1 + \sqrt{r_{uu}r_{dd}}} \right) \right].
\end{align*}
\]
Problem 2.3. A call on a bond

Using the BDT tree

\[ r_{uu} = 0.0727 \]
\[ r_u = 0.0568 \]
\[ r_0 = 0.04 \quad r_{ud} = 0.0595 \]
\[ r_d = 0.0483 \]
\[ r_{uu} = 0.0487 \]

let us find the price of a two-year, at-the-money call option on a three-year, zero-coupon bond redeemable for $1.

**Solution:** The call option is at the money. So, its strike is \( K = P(0, 3) = (1.05)^{-3} = 0.8638 \).

According to our tree, the three possible bond prices at time-2 (the call’s exercise date) are

\[ P_{uu} = (1.0727)^{-1} = 0.9322, \quad P_{ud} = (1.0595)^{-1} = 0.9438, \quad P_{dd} = (1.0487)^{-1} = 0.9536. \]

Note that the call option is always in the money at expiration. It is possible to save some time by calculating \( P_{uu} \) first and then noticing that the remaining possible bond prices must be higher which makes the call option even more in the money at nodes \( ud \) and \( dd \). Thus, the call option’s price is

\[ V_C(0) = \frac{1}{1 + r_0} \times \frac{1}{4} \times \left[ \frac{1}{1 + r_u} \left( (P_{uu} - K) + (P_{ud} - K) \right) + \frac{1}{1 + r_d} \left( (P_{ud} - K) + (P_{dd} - K) \right) \right] \]
\[ = \frac{1}{1 + r_0} \times \frac{1}{4} \times \left[ \frac{1}{1 + r_u} (P_{uu} + P_{ud}) + \frac{1}{1 + r_d} (P_{ud} + P_{dd}) \right] - \frac{1}{1 + r_0} \times \frac{1}{4} \times \left[ \frac{1}{1 + r_u} (-2K) + \frac{1}{1 + r_d} (-2K) \right] \]
\[ = P(0, 3) - KP(0, 2) = P(0, 3)(1 - P(0, 2)) = 0.8638(1 - 0.91573) = 0.072796. \]

Problem 2.4. BDT caplet pricing

Consider the following values of interest rates from an incomplete Black-Derman-Toy interest rate tree for the effective annual interest rates.

\[ r_0 = 0.09, \quad r_u = 0.12, \quad r_{uu} = 0.15, \]
\[ r_d = 0.08, \quad r_{ud} = 0.13. \]

(i) (2 points) Find the volatility \( \sigma_1 \) of the interest rates at time-1.

(ii) (3 points) Find the interest rate \( r_{dd} \) missing from the tree.

(iii) (5 points) Consider a 3-year caplet for the notional amount of $100 whose cap rate is given to be 11.5%. Calculate its price.

**Solution:**

(i)

\[ \sigma_1 = \frac{1}{2} \ln(0.12/0.08) = 0.20273. \]

(ii)

\[ r_{dd} = \frac{r_{ud}^2}{r_{uu}} = 0.13^2/0.15 = 0.112667. \]
(iii) \[100 \times \frac{1}{1.09} \times \frac{1}{4} \times \left[ \frac{1}{1.15} \times \frac{1}{1.12} (0.15 - 0.115) + \frac{1}{1.13} \left( \frac{1}{1.12} + \frac{1}{1.08} \right) (0.13 - 0.115) \right] = 1.177.\]

**Problem 2.5. Yield volatility**

In a Black-Derman-Toy tree, the annual effective interest rates (in our usual notation) are given to be
\[r_0, \quad r_u = 0.05, \quad r_{uu} = 0.06, \quad r_d = 0.045, \quad r_{dd} = 0.04.\]

(i) (5 points) Calculate \(r_{ud}\).

(ii) (15 points) Compute the “volatility in year 1” of the 3-year zero-coupon bond generated by the tree.

**Caveat:** This is not the volatility of the effective interest rates in the tree at any single year!

**Solution:** Note: Compare this problem to the Sample MFE Problem #29.

(i)
\[r_{ud} = \sqrt{r_{uu} \cdot r_{dd}} = \sqrt{0.04 \cdot 0.06} = 0.049.\]

(ii) From the perspective of the up node, the bond from the problem has the price
\[P_u = \frac{1}{1 + r_u} \times \frac{1}{2} \left[ \frac{1}{1 + r_{uu}} + \frac{1}{1 + r_{ud}} \right].\]
Similarly, from the perspective of the down node, the bond from the problem has the price
\[P_d = \frac{1}{1 + r_d} \times \frac{1}{2} \left[ \frac{1}{1 + r_{ud}} + \frac{1}{1 + r_{dd}} \right].\]

We know that, in the notation used in class,
\[P[h, T, r(h)] = (1 + y[h, T, r(h)])^{-(T-h)} .\]

In the present problem, the above general equality translates to
\[P[1, 3, r(1)] = (1 + y[1, 3, r(1)])^{-2}.\]

So, the bond’s yield to maturity from the perspective of the up node is
\[y_u = (P_u)^{-1/2} - 1.\]

In the same way, the bond’s yield from the perspective of the down node is
\[y_d = (P_d)^{-1/2} - 1.\]

Let us temporarily denote the volatility in year 1 of the 3-year zero-coupon bond generated by the tree by \(\kappa\). Then, by definition and recalling that every period in the above tree is one year long, \(\kappa\) must satisfy
\[y_u = y_d e^{2\kappa} \iff \kappa = \frac{1}{2} \ln \left( \frac{y_u}{y_d} \right).\]

In the present problem,
\[P_u = 0.9032 \quad \Rightarrow \quad y_u = 0.0522, \quad P_d = 0.9162 \quad \Rightarrow \quad y_d = 0.0447.\]

So,
\[\kappa = \frac{1}{2} \ln \left( \frac{y_u}{y_d} \right) = \frac{1}{2} \ln \left( \frac{0.0522}{0.0447} \right) = 0.0776.\]

**Instructor:** Milica Ćudina