**Problem 1.1.** (2 points) Consider the following Black-Derman-Toy tree.

Then, $r_{ud}$ is the geometric average of $r_{uu}$ and $r_{dd}$. True or false?

**Solution:** TRUE

**Problem 1.2.** (5 points) It is observed that the bond prices for zero coupon bonds redeemable for $1$ are, in our usual notation,

$$P(0,1) = 0.9615 \quad \text{and} \quad P(0,2) = 0.9157.$$ 

In the Black-Derman-Toy tree, the base rate parameter is set to equal $r_d = 0.035$. What is the interval into which the value of the volatility of effective interest rates in year-2 falls?

(a) $[0, 0.1)$
(b) $[0.1, 0.15)$
(c) $[0.15, 0.25)$
(d) $[0.25, 0.35)$
(e) None of the above.

**Solution:** (d)

We need to solve for the parameter $\sigma_1$ in the calibration equation. The calibration equation is

$$P(0,2) = P(0,1) \times \frac{1}{2} \left[ \frac{1}{1+r_d} + \frac{1}{1+r_d e^{2\sigma_1}} \right].$$

In the present problem, we have

$$0.9157 = 0.9615 \times \frac{1}{2} \left[ \frac{1}{1.035} + \frac{1}{1.035 e^{2\sigma_1}} \right]$$
So,

\[
1.9047 = \frac{1}{1.035} + \frac{1}{1 + 0.035e^{2\sigma_1}} \quad \Rightarrow \quad 1 + 0.035e^{2\sigma_1} = 1.0168 \quad \Rightarrow \quad \sigma_1 = 0.3134.
\]
Problem 1.3. (8 points) The following is a Black-Derman-Toy tree containing the prices of one-year, zero-coupon bonds (redeemable for $1) at intervals of length of one year:

Using the fact that this is a Black-Derman-Toy tree, calculate $P_{dd}$.

Solution:
In our usual notation, $P_{dd} = \frac{1}{1+r_{dd}}$. So, we need to first calculate $r_{dd}$ first from the given information. Since this is a Black-Derman-Toy tree,

$$r_{dd} = \frac{r_{ud}^2}{r_{uu}}$$

From the given bond prices, we get

$$r_{uu} = P_{uu}^{-1} - 1 = 0.945^{-1} - 1 = 0.0582,$$

$$r_{ud} = P_{ud}^{-1} - 1 = 0.956^{-1} - 1 = 0.04603.$$  

Hence,

$$r_{dd} = \frac{0.04603^2}{0.0582} = 0.0364.$$  

Finally, the bond price we were looking for is

$$P_{dd} = 1.0364^{-1} = 0.9649.$$