Notes: This is a closed book and closed notes exam.
Time: 50 minutes
1.1. **TRUE/FALSE QUESTIONS.** *Please note your answers on the front page.*

**Problem 1.1.** (2 pts) In our usual notation, the time—$t$ forward price of a bond deliverable at $T$ is

$$F_{t,T}[P(T, T + s)] = \frac{P(t, T + s)}{P(t, T)}.$$  

*Solution: TRUE*

**Problem 1.2.** (2 pts) A caplet is a financial instrument used as protection against the increase in the interest rate.

*Solution: TRUE*

**Problem 1.3.** (2 pts) The interest rate cap pays the difference (if it is positive) between the realized interest rate in a period and the cap rate on every date a loan repayment installment is to be made.

*Solution: TRUE*

**Problem 1.4.** (2 pts) In the Black-Derman-Toy model, the values at the two nodes coming from a node in the previous period are centered around the value at that (parent) node.

*Solution: FALSE*

**Problem 1.5.** (2 points) Let the future effective interest rates be modeled by a Black-Derman-Toy model. Then, in our usual notation:

$$r_d < r_0 < r_u.$$  

*Solution: FALSE*

**Problem 1.6.** In the Black-Derman-Toy model, the interest rate at any node is the geometric average of the rates at the two nodes at adjacent heights.

*Solution: TRUE*
1.2. **FREE-RESPONSE PROBLEMS.** Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they’re correct) are worth 0 points.

**Problem 1.7.** (20 points) The evolution of **continuously-compounded** annual interest rates is modeled by a two-step binomial interest-rate model. Your modeling assumptions are that:

- \( r_0 = 0.06; \)
- the interest rate can either increase or decrease by 0.02 in any single step;
- the (risk-neutral) probability of a step up is 1/2 for every single step.

A two-year European call option gives its bearer the right to purchase a zero-coupon bond with one year to maturity (when it will be redeemable for $1) for $0.92. What’s the price of the call option?

**Solution:** The possible time-2 bond prices are

\[
P_{uu} = e^{-10} = 0.9048, \quad P_{ud} = e^{-0.06} = 0.9418, \quad P_{dd} = 0.9802.\]

The call’s possible payoffs are, therefore,

\[
V_{uu} = 0, \quad V_{ud} = 0.0218, \quad V_{dd} = 0.0602.\]

The call’s price is

\[
V_C(0) = 0.25e^{-0.06}[V_{ud}(e^{-0.08} + e^{-0.04}) + V_{dd}e^{-0.04}] = 0.0233.\]

**Problem 1.8.** (20 points) Consider the following values of interest rates from an incomplete **Black-Derman-Toy** interest-rate tree for the effective annual interest rates.

\[
r_u = 0.3, \quad r_{uu} = 0.4, \quad r_d = 0.2, \quad r_{dd} = 0.1.\]

Let \( F \) denote the forward price for delivery at time-2 of a zero-coupon bond redeemable at time-3 for $1000. Calculate \( F \).

**Solution:** As we have seen in class, the above forward price can be obtained as

\[
F = 1000 \times \frac{P(0,3)}{P(0,2)}\]

where \( P(0,T) \) denotes the price of a zero-coupon bond redeemable at time-\( T \) for $1. Using the provided tree, we get

\[
P(0, 2) = \frac{1}{1 + r_0} \times \frac{1}{2} \times \left[ \frac{1}{1 + r_u} + \frac{1}{1 + r_d} \right]\]

and

\[
P(0, 3) = \frac{1}{1 + r_0} \times \frac{1}{4} \times \left[ \frac{1}{1 + r_u} \times \left( \frac{1}{1 + r_{uu}} + \frac{1}{1 + r_{ud}} \right) + \frac{1}{1 + r_d} \times \left( \frac{1}{1 + r_{dd}} + \frac{1}{1 + r_{ud}} \right) \right]
= \frac{1}{1 + r_0} \times \frac{1}{4} \times \left[ \frac{1}{1 + r_u} \times \left( \frac{1}{1 + r_{uu}} + \frac{1}{1 + \sqrt{r_{uu}r_{dd}}} \right) + \frac{1}{1 + r_d} \times \left( \frac{1}{1 + r_{dd}} + \frac{1}{1 + \sqrt{r_{uu}r_{dd}}} \right) \right].\]
So,

\[
F = 1000 \times \frac{1}{2} \times \left[ \frac{1}{1+r_u} \times \left( \frac{1}{1+r_{uu}} + \frac{1}{1+\sqrt{r_{uu}r_{dd}}} \right) + \frac{1}{1+r_d} \times \left( \frac{1}{1+r_{dd}} + \frac{1}{1+\sqrt{r_{uu}r_{dd}}} \right) \right] \frac{1}{1+r_u} + \frac{1}{1+r_d}
\]

\[
= 1000 \times \frac{1}{2} \times \left[ \frac{1.13}{1.1} \times \left( \frac{1.14}{1.4 + 1.04} \right) + \frac{1.12}{1.2} \times \left( \frac{1.12}{1.2 + 1.04} \right) \right] \frac{1}{1.3} + \frac{1}{1.2}
\]

\[
= 824.46
\]

**Problem 1.9.** (10 points)

The evolution of effective annual interest rates is modeled by a three-period tree so that during each period the interest rate can either increase by 0.02 or decrease by 0.01. Let the initial (root) interest rate be equal to 0.08. The risk-neutral probability of an up movement is 0.6 at every node.

Consider a three-year interest-rate cap with a cap rate equal to 0.09 purchased to hedge against adverse movements of the above-modeled floating rate. Let the loan amount be equal to $1000 and let the loan repayment installments be interest-only.

What is the price of the above cap consistent with our model?

**Solution:** The nodes in the tree at which the cap makes a positive payment, along with the respective payment amounts are:

\[
up : 1000(0.10 - 0.09) = 1000(0.01),
\]

\[
up - up : 1000(0.12 - 0.09) = 1000(0.03).
\]

So, the cap’s price equals

\[
1000 \times \frac{1}{1.08} \left[ 0.6 \times \frac{0.01}{1.1} + (0.6)^2 \times \frac{0.03}{1.12(1.1)} \right] = \frac{1000(0.6)(0.01)}{1.08(1.1)} \left[ 1 + \frac{0.6(3)}{1.12} \right]
\]

\[
= \frac{6}{1.08(1.1)}(2.607143) = 13.16739
\]

**Problem 1.10.** (20 points)

A three-period Black-Derman-Toy tree is calibrated so that

\[
\begin{align*}
r_0 & = 0.03, r_u = 0.04, r_{uu} = 0.05, \\
r_d & = 0.035, r_{ud} = 0.04.
\end{align*}
\]

Consider a one-year, $0.93-strike put option on a zero-coupon bond which matures at time $3 for $1. What is the price of this put option consistent with the above interest-rate model?

**Solution:** The bond prices at the up and down nodes are

\[
P_u = \frac{1}{1.04} \times \frac{1}{2} \left[ \frac{1}{1.05} + \frac{1}{1.04} \right] = 0.92015.
\]

\[
P_d = \frac{1}{1.035} \times \frac{1}{2} \left[ \frac{1}{1.032} + \frac{1}{1.04} \right] = 0.9326.
\]
So, the price of the put is

\[ V_P(0) = \frac{1}{1.03} \times \frac{1}{2} (0.93 - 0.92015) = 0.004126214. \]

1.3. MULTIPLE CHOICE QUESTIONS. Please, record your answers on the front page of this exam.

**Problem 1.11.** (5 points) The price of a zero-coupon bond redeemable in one year equals $93.50, while the price of a zero-coupon bond redeemable in two years equals $82.50.

You are using the above bond prices to calibrate a Black-Derman-Toy tree of effective annual interest rates under the additional assumption that the volatility of interest rates in the second period equals 0.09.

Let \( r_d \) denote the interest rate in the “down” state. Then, \( r_d \) falls within the following interval:

(a) [0, 0.07)
(b) [0.07, 0.09)
(c) [0.09, 0.12)
(d) [0.12, 0.14)
(e) None of the above.

**Solution:** (d)

We need to solve for \( r_d \) in

\[
82.50 = 93.50 \times \frac{1}{2} \times \left( \frac{1}{1 + r_d e^{0.18}} + \frac{1}{1 + r_d} \right).
\]

Simplifying the above equation, we get this quadratic in \( r_d \)

\[ 2.11r_d^2 + 1.68r_d - 0.24 = 0. \]

So, \( r_d = 0.1237 \).

**Problem 1.12.** (5 points) Your task is to find the forward price for delivery two years from now of a zero-coupon bond redeemable for $1000 three years from now. To model effective annual interest rates, you use the Black-Derman-Toy model with

\[
\begin{align*}
  r_u &= 0.40, & r_{uu} &= 0.45 \\
  r_d &= 0.25, & r_{dd} &= 0.20.
\end{align*}
\]

(a) 757.30
(b) 767.40
(c) 857.50
(d) 876.60
(e) None of the above.

**Solution:** (b)
Problem 1.13. (5 points) The evolution of effective annual interest rates over the following three one-year periods is modeled by the following tree:

Assume that the risk-neutral probability of an up movement in a single period equals 1/2.

Consider a two-year, $95-strike European call option on a zero-coupon bond redeemable for $100 three years from today. What is the price of this call option?

(a) $0.69
(b) $0.73
(c) $0.98
(d) $1.38
(e) None of the above.

Solution: (a)
The three possible bond prices on the call’s exercise date are

\[ P_{uu} = \frac{1}{1.10} = 0.9090, \quad P_{ud} = \frac{1}{1.06} = 0.9434, \quad P_{dd} = \frac{1}{1.02} = 0.9804. \]

So, the only final state of the world in which there is a non-zero payoff is the \( dd \) node. There, the call’s payoff equals

\[ V_{dd} = 98.04 - 95 = 3.04. \]
Finally, the price of the call today equals

\[ V_C(0) = \frac{1}{1.06} \times \frac{1}{4} \times \frac{1}{1.04} \times 3.04 = 0.6894. \]
Problem 1.14. (5 points)
The two-period interest-rate tree models effective annual interest rates. The two possible interest rates for the time period \([1, 2]\) are given to be 0.04 and 0.02. The risk-neutral probability of an up move is specified as 0.55.

The price of a zero-coupon bond redeemable in two years for $1 is $0.95.

What is the root effective interest rate consistent with the above bond prices?

(a) About 0.021.
(b) About 0.023.
(c) About 0.024.
(d) About 0.025.
(e) None of the above.

Solution: (a)

\[
0.95 = P(0, 1) \left[ \frac{0.55}{1.04} + \frac{0.45}{1.02} \right] \quad \Rightarrow \quad r_0 = 0.021.
\]