Notes: This is a closed book and closed notes exam.
Time: 50 minutes

MULTIPLE CHOICE

TRUE/FALSE

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<thead>
<tr>
<th>1 (2)</th>
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<td>2 (2)</td>
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1 (5) 2 (5) 3 (5) 4 (5) 5 (5) 6 (5)

FOR GRADER'S USE ONLY:

T/F 1. 2. M.C. Σ
2.1. **TRUE/FALSE QUESTIONS.** Please note your answers on the front page.

**Problem 2.1.** Continuously compounded returns of stocks are multiplicative.

**Solution:** FALSE

**Problem 2.2.** Let $X$ be a strictly positive random variable, then

$$E[X] = |E[X]|.$$

**Solution:** TRUE

**Problem 2.3.** (2 points)

Let $X$ be a random variable whose variance is zero. Then $X$ is constant with probability one.

**Solution:** TRUE

**Problem 2.4.** In the lognormal stock-price model, the mean time—$t$ stock price is increasing as a function of $t$.

**Solution:** FALSE

2.2. **FREE-RESPONSE PROBLEMS.** Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they’re correct) are worth 0 points.

**Problem 2.5.** (10 points) Let the stock prices be modeled using the lognormal distribution. The mean stock price at time—1 equals 120 and the median stock price 115. What is the probability that the time—1 stock price exceeds 100?

**Solution:** The stock price at time—1 is lognormally distributed. In fact, using our usual parameters, we can rewrite it as

$$S(1) = S(0)e^{(\alpha-\delta-\frac{1}{2}\sigma^2)+\sigma Z(1)}.$$

Recall that the median of $S(1)$ equals $S(0)e^{(\alpha-\delta-\frac{1}{2}\sigma^2)}$. So, the required probability can be expressed as

$$P[S(1) > 100] = P[115e^{\sigma Z(1)} > 100] = P\left[Z(1) > \frac{1}{\sigma} \ln\left(\frac{100}{115}\right)\right]$$

$$= P\left[Z(1) < \frac{1}{\sigma} \ln\left(\frac{115}{100}\right)\right] = N\left(\frac{1}{\sigma} \ln\left(\frac{115}{100}\right)\right).$$

Since the mean of $S(1)$ equals $S(0)e^{(\alpha-\delta)}$, we have

$$e^{\frac{1}{2}\sigma^2} = \frac{120}{115} \Rightarrow \sigma = \sqrt{2 \ln(1.04348)} = 0.2918.$$
So, our final answer is
\[ P[S(1) > 100] = N(0.48) = 0.6844. \]

2.3. **MULTIPLE CHOICE QUESTIONS.**

**Problem 2.6.** A continuous-dividend-paying stock has a dividend yield of 0.02. The time-\( t \) realized (rate of) return is modeled as
\[ R(0, t) \sim N(\text{mean} = 0.03t, \text{variance} = 0.09t) \]

Find the probability that the time-4 stock price exceeds today’s stock price.

(a) 0.4920  
(b) 0.4960  
(c) 0.5040  
(d) 0.5080  
(e) None of the above.

**Solution:** (e)

We need to find
\[ P[S(4) > S(0)] \]

with
\[ S(t) = S(0)e^{R(0,t)}. \]

Since \( R(0, t) \) follows the normal distribution with the above parameters, we have
\[
P[S(4) > S(0)] = P[S(0)e^{R(0,4)} > S(0)] = P[R(0, 4) > 0]
\]
\[ = 1 - N \left( -\frac{0.03 \times 4}{0.3 \times 2} \right) = N(0.2) = 0.5793. \]
Problem 2.7. The current price of continuous-dividend-paying stock is $100 per share and its volatility is 0.25. The stock price is modeled using a log-normal distribution. The expected time-2 stock price is $120.

Then, the median of the time-2 stock price falls within this interval:

(a) [0, 86)
(b) [86, 106)
(c) [106, 112)
(d) [112, 124)
(e) None of the above.

Solution: (d)
The median of the time-2 stock price is
\[
\mathbb{E}[S(2)]e^{-\frac{\sigma^2}{2}} = 120e^{-\frac{0.0625}{2}} = 120e^{-0.03125} \approx 116.31.
\]

Problem 2.8. (5 pts)
A non-dividend-paying stock is currently valued at $100 per share. Its annual mean rate of return is given to be 12% while its volatility is given to be 30%.

Assuming the lognormal stock-price model, find
\[
\mathbb{E}[S(2) \mid S(2) > 95].
\]

(a) $86.55
(b) $101.60
(c) $152.35
(d) $159.07
(e) None of the above.

Solution: (c)
In our usual notation,
\[
\mathbb{E}[S(T) \mid S(T) > K] = \frac{S(0)e^{(\alpha - \delta)T}N(\hat{d}_1)}{N(\hat{d}_2)},
\]
with
\[
\hat{d}_1 = \frac{1}{\sigma \sqrt{T}} \left[ \ln \left( \frac{S(0)}{K} \right) + \left( \alpha - \delta + \frac{\sigma^2}{2} \right) T \right],
\]\[
\hat{d}_2 = \hat{d}_1 - \sigma \sqrt{T}.
\]

In the present problem,
\[
\hat{d}_1 = \frac{1}{0.3\sqrt{2}} \left[ \ln \left( \frac{100}{95} \right) + \left( 0.12 + \frac{0.09}{2} \right) \times 2 \right] = 0.8987,
\]\[
\hat{d}_2 = 0.8987 - 0.3\sqrt{2} = 0.4745.
\]
So, our answer is
\[ E[S(2) | S(2) > 95] = \frac{100e^{(0.12)\times2}N(0.8987)}{N(0.4745)} = \frac{100e^{0.24} \times 0.8159}{0.6808} = 152.35. \]

**Problem 2.9.** (5 pts) Assume the Black-Scholes framework. Let the current price of a non-dividend-paying stock be equal to \( S(0) = 95 \) and let its volatility be equal to 0.35. Consider a European call on that stock with strike 100 and exercise date in 9 months. Let the risk-free continuously compounded interest rate be 6% per annum.

Denote the price of the call by \( V_C(0) \). Then,

(a) \( V_C(0) < 5.20 \)
(b) \( 5.20 \leq V_C(0) < 7.69 \)
(c) \( 7.69 \leq V_C(0) < 9.04 \)
(d) \( 9.04 \leq V_C(0) < 11.25 \)
(e) None of the above.

**Solution:** (d)
Using the Black-Scholes formula one gets the price of about 11.06.

**Problem 2.10.** (5 pts) Assume the Black-Scholes framework. Let the current price of a share of stock be equal to \( S(0) = 80 \), let its volatility be \( \sigma = 0.3 \), and let \( \delta = 0 \) (in our usual notation).

Consider a gap option with expiration date \( T = 1 \) year such that its payoff is \( S(T) - 90 \) if \( S(T) > 100 \).

You are given that the continuously compounded risk-free interest rate equals \( r = 0.05 \) per annum.

Let \( V_{GC}(0) \) denote the price of the above gap option. Then,

(a) \( V_{GC}(0) < 3.20 \)
(b) \( 3.20 \leq V_{GC}(0) < 5.69 \)
(c) \( 5.69 \leq V_{GC}(0) < 7.04 \)
(d) \( 7.04 \leq V_{GC}(0) < 11.25 \)
(e) None of the above.

**Solution:** (c)
In our usual notation, the Black-Scholes formula for the price of a gap call option reads as
\[
V_{GC}(0) = S(0)^{-\delta T}N(d_1) - K_1e^{-rT}N(d_2)
\]
where
\[
d_1 = \frac{1}{\sigma\sqrt{T}}[\ln(S(0)/K_2) + (r - \frac{1}{2}\sigma^2)T],
\]
\[
d_2 = d_1 - \sigma\sqrt{T}.
\]
In the present problem,
\[
d_1 = \frac{1}{0.3}[\ln(80/100) + (0.05 + \frac{0.09}{2})] \approx -0.43,
\]
\[
d_2 = -0.73.
\]
So, the price equals
\[ V_{GC}(0) = 80N(-0.43) - 90e^{-0.05}N(-0.73) = 80 \cdot (1 - 0.6664) - 90e^{-0.05} \cdot (1 - 0.7673) = 6.7664. \]

**Problem 2.11.** Assume the Black-Scholes setting.
Mary wagers to pay one share of stock to Matt if the price at expiration in 1 year is above $75.00. Assume \( S(0) = 60.00, \sigma = 0.15, r = 0.04, \) and the dividend rate of 0.01. What is the value of Mary’s bet?
(a) 6.72
(b) 7.52
(c) 8.72
(d) 9.51
(e) None of the above.

**Solution:** (a)
This is the Black-Scholes price of an asset call, i.e.,
\[ V_{AC}(0) = S(0)e^{-0.01} N(d_1) \]
with
\[ d_1 = \frac{1}{0.15} \left( \ln \left( \frac{60}{75} \right) + 0.04 - 0.01 + \frac{1}{2} 0.15^2 \right) = -1.21. \]
So,
\[ V_{AC}(0) = 60e^{-0.01} N(-1.21) = 60e^{-0.01} (1 - 0.8869) \approx 6.72. \]

**Problem 2.12.** Assume the Black-Scholes setting.
Assume \( S(0) = 63.75, \sigma = 0.20, r = 0.055. \) The stock pays no dividend and the option expires in 50 days (simplify the number of days in a year to 360).
What is the price of a $60-strike European put?
(a) 0.66
(b) 0.55
(c) 0.44
(d) 0.37
(e) None of the above.

**Solution:** (d)
In our usual notation, the price is
\[ V_P(0) = Ke^{-rT} N(-d_2) - S(0)N(-d_1) \]
with
\[ d_1 = \frac{1}{0.2\sqrt{5/36}} \left( \ln \left( \frac{63.75}{60} \right) + (0.055 + \frac{1}{2} 0.2^2) \left( \frac{5}{36} \right) \right) = 0.95, \]
\[ d_2 = d_1 - 0.25\sqrt{0.125} = 0.88. \]
So,
\[ V_P(0) = 60e^{-0.055 \cdot \frac{5}{36}} (1 - 0.8106) - 63.75 \cdot (1 - 0.8289) = 0.37. \]

**Problem 2.13.** Assume the Black-Scholes setting.
Assume \( S(0) = 28.50, \sigma = 0.32, r = 0.04. \) The stock pays a 1.0\% continuous dividend and the option expires in 110 days (simplify the number of days in a year to 360).
What is the price of a $30-strike put?

(a) 2.75  
(b) 2.10  
(c) 1.80  
(d) 1.20  
(e) None of the above.

**Solution:** (a)
In our usual notation, the price is
\[ V_P(0) = Ke^{-rT}N(-d_2) - S(0)e^{-\delta T}N(-d_1) \]
with
\[ d_1 = -0.15, \quad d_2 = -0.33. \]
So, \( V_P(0) = 2.75. \)

**Problem 2.14.** (5 points) Consider a non-dividend-paying stock whose price \( S = \{S(t), t \geq 0\} \) is modeled using a geometric Brownian motion. Suppose that the current stock price equals $100 and that its volatility is given to be 0.25.
The continuously compounded, risk-free interest rate is assumed to equal 0.04.
Consider a derivative security which entitles its owner to obtain a European call option on the above stock six months from today, i.e., at time \( t_*=1/2. \) The call option is to be 3-month to expiration at time of delivery and have the strike equal to 105\% of the time-\( t^* \) price of the underlying asset. This contract is called a **forward start option**.
What is the price of the forward start option?

(a) 3.15  
(b) 8.65  
(c) 10.51  
(d) 13.55  
(e) None of the above.

**Solution:** (a)
At time \( t^* \), the required Black-Scholes price of the call option equals
\[
V_C(t^*) = S(t^*)N(d_1) - 1.05S(t^*)e^{-r(T-t^*)}N(d_2) = S(t^*)(N(d_1) - 1.05e^{-0.01}N(d_2))
\]
with
\[ d_1 = \frac{1}{0.125} \left[ -\ln(1.05) + \left( 0.04 + \frac{0.25^2}{2} \right) \times \frac{1}{4} \right] = -0.2478, \]
\[ d_2 = d_1 - \sigma \sqrt{T - t^*} = -0.3728. \]
Hence,
\[ V_C(t^*) = S(t^*) (0.4013 - 1.05e^{-0.01} \times 0.3557) = \]
So, one would need to buy 0.0315 shares of stock to be able to buy the call option in question at time\(-t^*\). This amount of shares costs $3.15.