Notes: This is a closed book and closed notes exam.
Time: 50 minutes

TRUE/FALSE

| 1 (2) | TRUE   | FALSE |
| 2 (2) | TRUE   | FALSE |
| 3 (2) | TRUE   | FALSE |
| 4 (2) | TRUE   | FALSE |
| 5 (2) | TRUE   | FALSE |

MULTIPLE CHOICE

| 1 (5) | a | b | c | d | e |
| 2 (5) | a | b | c | d | e |
| 3 (5) | a | b | c | d | e |
| 4 (5) | a | b | c | d | e |
| 5 (5) | a | b | c | d | e |

FOR GRADER’S USE ONLY:

| T/F  | 1. | 2. | M.C. | Σ   |
2.1. **TRUE/FALSE QUESTIONS.** Please note your answers on the front page.

**Problem 2.1.** You are trying to estimate the mean rate of return on a stock. Then, using more frequent observations of the stock price produces a more accurate estimate.

**Solution:** FALSE

**Problem 2.2.** Continuously compounded returns of stocks are multiplicative.

**Solution:** FALSE

**Problem 2.3.** Let $X$ be a strictly positive random variable, then

$$E[X] = |E[X]|.$$

**Solution:** TRUE

**Problem 2.4.** (2 points)

Let $X$ be a random variable whose variance is zero. Then $X$ is constant with probability one.

**Solution:** TRUE

**Problem 2.5.** In the lognormal stock-price model, the mean time–$t$ stock price is increasing as a function of $t$.

**Solution:** FALSE

2.2. **FREE-RESPONSE PROBLEMS.** Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they’re correct) are worth 0 points.
Problem 2.6. (10 points)
We wish to simulate the time−4 price of a stock modeled using the lognormal distribution. In our usual notation,

1. the stock’s initial price is \( S(0) = 100 \),
2. the stock’s mean rate of return is \( \alpha = 0.12 \),
3. the continuously compounded risk-free interest rate is 0.04,
4. the stock’s dividend yield is 0.02,
5. the stock’s volatility is 0.25.

We draw a single unit uniform simulated value and get 0.2946. Which simulated value of the time−4 stock price do we get using the usual procedure, i.e., if we first use the inverse transformation method to get the normal draws?

Solution: Using the inverse transformation method, we get the following draw from the standard normal distribution

\[ N^{-1}(0.2946) = -0.54. \]

So, the simulated value of the stock price at time−2 is

\[ s = 100 e^{(0.12−0.02−\frac{1}{2} \cdot 0.25^2) \cdot 4 + 0.25 \sqrt{4}(-0.54)} = 100.50. \]

The simulated value of the time−4 stock price equals:
Problem 2.7. (10 points)
You observe the following stock prices in the beginning of every month:

<table>
<thead>
<tr>
<th>Observation</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock price</td>
<td>100</td>
<td>80</td>
<td>64</td>
<td>80</td>
<td>80</td>
<td>100</td>
</tr>
</tbody>
</table>

i. (5 points) Calculate the (unbiased) estimate of the annual volatility based on the above data.

ii. (5 points) Assuming that the stock-prices are modeled using a geometric Brownian motion, what is your estimate of the stock’s rate of appreciation?

Solution:

i. The initial and the final stock prices are both 100, so the observed average of the log-ratios is 0. The log-ratios based on the above data are

<table>
<thead>
<tr>
<th>$r_1$</th>
<th>$r_2$</th>
<th>$r_3$</th>
<th>$r_4$</th>
<th>$r_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ln \left( \frac{4}{5} \right)$</td>
<td>$\ln \left( \frac{4}{5} \right)$</td>
<td>$-\ln \left( \frac{4}{5} \right)$</td>
<td>0</td>
<td>$-\ln \left( \frac{4}{5} \right)$</td>
</tr>
</tbody>
</table>

So, the unbiased estimate of the volatility squared (on the monthly scale) is

$$\frac{1}{5-1} \times 4 \times \left( \ln \left( \frac{4}{5} \right) \right)^2$$

The annualized squared volatility estimate is

$$\hat{\sigma}^2 = 12 \times \left( \ln \left( \frac{4}{5} \right) \right)^2.$$

Finally, $\hat{\sigma} = 0.773$.

ii. Recall that the observed average of the log-ratios of stock prices equals zero. So,

$$\hat{\alpha} - \hat{\delta} = \frac{1}{2} \hat{\sigma}^2 = 0.2988.$$
Problem 2.8. (10 points) Let the stock prices be modeled using the lognormal distribution. The mean stock price at time $-1$ equals 120 and the median stock price 115. What is the probability that the time $-1$ stock price exceeds 100?

Solution: The stock price at time $-1$ is lognormally distributed. In fact, using our usual parameters, we can rewrite it as

$$S(1) = S(0)e^{(\alpha - \delta - \frac{1}{2}\sigma^2) + \sigma Z(1)}.$$ 

Recall that the median of $S(1)$ equals $S(0)e^{(\alpha - \delta - \frac{1}{2}\sigma^2)}$. So, the required probability can be expressed as

$$\mathbb{P}[S(1) > 100] = \mathbb{P}[115e^{\sigma Z(1)} > 100] = \mathbb{P}\left[Z(1) > \frac{1}{\sigma} \ln\left(\frac{100}{115}\right)\right] = \mathbb{P}\left[Z(1) < \frac{1}{\sigma} \ln\left(\frac{115}{100}\right)\right] = N\left(\frac{1}{\sigma} \ln\left(\frac{115}{100}\right)\right).$$

Since the mean of $S(1)$ equals $S(0)e^{(\alpha - \delta)}$, we have

$$e^{\frac{1}{2}\sigma^2} = \frac{120}{115} \Rightarrow \sigma = \sqrt{2 \ln(1.04348)} = 0.2918.$$ 

So, our final answer is

$$\mathbb{P}[S(1) > 100] = N(0.48) = 0.6844.$$
2.3. **MULTIPLE CHOICE QUESTIONS.**

**Problem 2.9.** A continuous-dividend-paying stock has a dividend yield of 0.02. The time—$t$ realized (rate of) return is modeled as

\[ R(0, t) \sim N(\text{mean} = 0.03t, \text{variance} = 0.09t) \]

Find the probability that the time—4 stock price exceeds today’s stock price.

(a) 0.4920  
(b) 0.4960  
(c) 0.5040  
(d) 0.5080  
(e) None of the above.

**Solution:** (e)

We need to find

\[ P[S(4) > S(0)] \]

with

\[ S(t) = S(0)e^{R(0,t)}. \]

Since $R(0, t)$ follows the normal distribution with the above parameters, we have

\[ P[S(4) > S(0)] = P[S(0)e^{R(0,4)} > S(0)] = P[R(0, 4) > 0] \]

\[ = 1 - N \left( -\frac{0.03 \times 4}{0.3 \times 2} \right) = N(0.2) = 0.5793. \]
Problem 2.10. The current price of continuous-dividend-paying stock is $100 per share and its volatility is 0.25. The stock price is modeled using a log-normal distribution. The expected time−2 stock price is $120.

Then, the median of the time−2 stock price falls within this interval:
(a) [0, 86)
(b) [86, 106)
(c) [106, 112)
(d) [112, 124)
(e) None of the above.

Solution: (d)
The median of the time−2 stock price is
\[
\mathbb{E}[S(2)]e^{-\frac{\sigma^2}{2}} = 120e^{-\frac{0.0625}{2}} = 120e^{-0.03125} \approx 116.31.
\]

Problem 2.11. (5 pts)
A non-dividend-paying stock is currently valued at $100 per share. Its annual mean rate of return is given to be 12% while its volatility is given to be 30%.

Assuming the lognormal stock-price model, find
\[
\mathbb{E}[S(2) \mid S(2) > 95].
\]
(a) $86.55
(b) $101.60
(c) $152.35
(d) $159.07
(e) None of the above.

Solution: (c)
In our usual notation,
\[
\mathbb{E}[S(T) \mid S(T) > K] = \frac{S(0)e^{(\alpha - \delta)T}N(\hat{d}_1)}{N(\hat{d}_2)}.
\]
with
\[
\hat{d}_1 = \frac{1}{\sigma \sqrt{T}} \left[ \ln \left( \frac{S(0)}{K} \right) + \left( \alpha - \delta + \frac{\sigma^2}{2} \right) T \right],
\]
\[
\hat{d}_2 = \hat{d}_1 - \sigma \sqrt{T}.
\]
In the present problem,
\[
\hat{d}_1 = \frac{1}{0.3 \sqrt{2}} \left[ \ln \left( \frac{100}{95} \right) + \left( 0.12 + \frac{0.09}{2} \right) \times 2 \right] = 0.8987,
\]
\[
\hat{d}_2 = 0.8987 - 0.3 \sqrt{2} = 0.4745.
\]
So, our answer is
\[ E[S(2) \mid S(2) > 95] = \frac{100e^{(0.12) \times 2}N(0.8987)}{N(0.4745)} = \frac{100e^{0.24} \times 0.8159}{0.6808} = 152.35. \]

**Problem 2.12.** (5 pts) Assume the Black-Scholes framework. Let the current price of a non-dividend-paying stock be equal to \( S(0) = 95 \) and let its volatility be equal to 0.35. Consider a European call on that stock with strike 100 and exercise date in 9 months. Let the risk-free continuously compounded interest rate be 6% per annum.

Denote the price of the call by \( V_C(0) \). Then,

(a) \( V_C(0) < 5.20 \)
(b) \( 5.20 \leq V_C(0) < 7.69 \)
(c) \( 7.69 \leq V_C(0) < 9.04 \)
(d) \( 9.04 \leq V_C(0) < 11.25 \)
(e) None of the above.

**Solution:** (d)
Using the Black-Scholes formula one gets the price of about 11.06.

**Problem 2.13.** (5 pts) Assume the Black-Scholes framework. Let the current price of a share of stock be equal to \( S(0) = 80 \), let its volatility be \( \sigma = 0.3 \), and let \( \delta = 0 \) (in our usual notation).

Consider a gap option with expiration date \( T = 1 \) year such that its payoff is \( S(T) - 90 \) if \( S(T) > 100 \).

You are given that the continuously compounded risk-free interest rate equals \( r = 0.05 \) per annum.

Let \( V_{GC}(0) \) denote the price of the above gap option. Then,

(a) \( V_{GC}(0) < 3.20 \)
(b) \( 3.20 \leq V_{GC}(0) < 5.69 \)
(c) \( 5.69 \leq V_{GC}(0) < 7.04 \)
(d) \( 7.04 \leq V_{GC}(0) < 11.25 \)
(e) None of the above.

**Solution:** (c)
In our usual notation, the Black-Scholes formula for the price of a gap call option reads as

\[ V_{GC}(0) = S(0)^{-\delta T}N(d_1) - K_1e^{-rT}N(d_2) \]

where

\[ d_1 = \frac{1}{\sigma \sqrt{T}}[\ln(S(0)/K_2) + (r - \delta + \frac{1}{2}\sigma^2)T], \]

\[ d_2 = d_1 - \sigma \sqrt{T}. \]

In the present problem,

\[ d_1 = \frac{1}{0.3}[\ln(80/100) + (0.05 + \frac{0.09}{2})] \approx -0.43, \]

\[ d_2 = -0.73. \]
So, the price equals
\[ V_{GC}(0) = 80N(-0.43) - 90e^{-0.05}N(-0.73) = 80 \cdot (1 - 0.6664) - 90e^{-0.05} \cdot (1 - 0.7673) = 6.7664. \]

**Problem 2.14.** Assume the Black-Scholes setting.
Mary wagers to pay one share of stock to Matt if the price at expiration in 1 year is above $75.00. Assume \( S(0) = 60.00, \sigma = 0.15, r = 0.04 \), and the dividend rate of 0.01. What is the value of Mary’s bet?
(a) 6.72
(b) 7.52
(c) 8.72
(d) 9.51
(e) None of the above.

**Solution:** (a)
This is the Black-Scholes price of an asset call, i.e.,
\[ V_{AC}(0) = S(0)e^{-0.01} N(d_1) \]
with
\[ d_1 = \frac{1}{0.15} \left( \ln \left( \frac{60}{75} \right) + 0.04 - 0.01 + \frac{1}{2} 0.15^2 \right) = -1.21. \]
So,
\[ V_{AC}(0) = 60e^{-0.01} N(-1.21) = 60e^{-0.01} (1 - 0.8869) \approx 6.72. \]

**Problem 2.15.** Assume the Black-Scholes setting.
Assume \( S(0) = $63.75, \sigma = 0.20, r = 0.055 \). The stock pays no dividend and the option expires in 50 days (simplify the number of days in a year to 360). What is the price of a $60-strike European put?
(a) 0.66
(b) 0.55
(c) 0.44
(d) 0.37
(e) None of the above.

**Solution:** (d)
In our usual notation, the price is
\[ V_P(0) = Ke^{-rT} N(-d_2) - S(0)N(-d_1) \]
with
\[ d_1 = \frac{1}{0.2 \sqrt{5/36}} \left( \ln \left( \frac{63.75}{60} \right) + (0.055 + \frac{1}{2} 0.2^2) \left( \frac{5}{36} \right) \right) = 0.95, \]
\[ d_2 = d_1 - 0.25 \sqrt{0.125} = 0.88. \]
So,

\[ V_P(0) = 60e^{-0.055 \cdot \frac{5}{36}} (1 - 0.8106) - 63.75 \cdot (1 - 0.8289) = 0.37. \]

**Problem 2.16.** Assume the Black-Scholes setting.
Assume \(S(0) = 28.50, \sigma = 0.32, r = 0.04\). The stock pays a 1.0% continuous dividend and the option expires in 110 days (simplify the number of days in a year to 360).
What is the price of a $30-strike put?

(a) 2.75
(b) 2.10
(c) 1.80
(d) 1.20
(e) None of the above.

**Solution:** (a)

In our usual notation, the price is

\[ V_P(0) = Ke^{-rT}N(-d_2) - S(0)e^{-\delta T}N(-d_1) \]

with

\[ d_1 = -0.15, \quad d_2 = -0.33. \]

So, \(V_P(0) = 2.75\).

**Problem 2.17.** (5 points) Consider a non-dividend-paying stock whose price \(S = \{S(t), t \geq 0\}\) is modeled using a geometric Brownian motion. Suppose that the current stock price equals $100 and that its volatility is given to be 0.25.
The continuously compounded, risk-free interest rate is assumed to equal 0.04.

Consider a derivative security which entitles its owner to obtain a European call option on the above stock six months from today, i.e., at time \(t_s = 1/2\). The call option is to be 3-month to expiration at time of delivery and have the strike equal to 105% of the time-\(t_s\) price of the underlying asset. This contract is called a forward start option.

What is the price of the forward start option?

(a) 3.15
(b) 8.65
(c) 10.51
(d) 13.55
(e) None of the above.

**Solution:** (a)

At time \(t_s\), the required Black-Scholes price of the call option equals

\[
V_C(t^*) = S(t^*)N(d_1) - 1.05S(t^*)e^{-r(T-t^*)}N(d_2)
= S(t^*)(N(d_1) - 1.05e^{-0.01}N(d_2))
\]
with
\[
d_1 = \frac{1}{0.125} \left[ -\ln(1.05) + \left( 0.04 + \frac{0.25^2}{2} \right) \times \frac{1}{4} \right] = -0.2478, \\
d_2 = d_1 - \sigma \sqrt{T - t^*} = -0.3728.
\]

Hence,
\[
V_C(t^*) = S(t^*)(0.4013 - 1.05e^{-0.01} \times 0.3557) = 
\]
So, one would need to buy 0.0315 shares of stock to be able to buy the call option in question at time\( -t^*\). This amount of shares costs $3.15.