On Market-Making and Delta-Hedging

1 Market Makers

2 Market-Making and Bond-Pricing
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2. Market-Making and Bond-Pricing
What to market makers do?

- Provide **immediacy** by standing ready to sell to buyers (at ask price) and to buy from sellers (at bid price)
- Generate **inventory** as needed by short-selling
- **Profit** by charging the bid-ask spread
- Their position is determined by the order flow from customers
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Market Maker Risk

- Market makers attempt to hedge in order to avoid the risk from their arbitrary positions due to customer orders (see Table 13.1 in the textbook)
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Delta and Gamma as measures of exposure

- Suppose that Delta is 0.5824, when $S = 40$ (same as in Table 13.1 and Figure 13.1)
- A $0.75$ increase in stock price would be expected to increase option value by $0.4368$ (increase in price $\times$ Delta $= 0.75 \times 0.5824$)
- The actual increase in the options value is higher: $0.4548$
- This is because the Delta increases as stock price increases. Using the smaller Delta at the lower stock price understates the actual change
- Similarly, using the original Delta overstates the change in the option value as a response to a stock price decline
- Using Gamma in addition to Delta improves the approximation of the option value change (Since Gamma measures the change in Delta as the stock price varies - it's like adding another term in the Taylor expansion)
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• We will begin by seeing how the Black model can be used to price bond and interest rate options
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A bond portfolio manager might want to hedge bonds of one duration with bonds of a different duration. This is called duration hedging. In general, hedging a bond portfolio based on duration does not result in a perfect hedge.

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The Dynamics of Bonds and Interest Rates

- Suppose that the bond-price at time \( T - t \) before maturity is denoted by \( P(t, T) \) and that it is modeled by the following Ito process:

\[
\frac{dP_t}{P_t} = \alpha(r, t) \, dt + q(r, t) \, dZ_t
\]

where

1. \( Z \) is a standard Brownian motion
2. \( \alpha \) and \( q \) are coefficients which depend both on time \( t \) and the interest rate \( r \)

- This approach requires careful specification of the coefficients \( \alpha \) and \( q \) - and we would like for the model to be simpler ...

- The alternative is to start with the model of the short-term interest rate as an Ito process:

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dr = a(r) \, dt + \sigma(r) \, dZ
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An Inappropriate Bond-Pricing Model

- We need to be careful when implementing the above strategy.
- For instance, if we assume that the yield-curve is flat, i.e., that at any time the zero-coupon bonds at all maturities have the same yield to maturity, we get that there is possibility for arbitrage.
- The construction of the portfolio which creates arbitrage is similar to the one for different Sharpe Ratios and a single source of uncertainty. You should read Section 24.1.
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An Equilibrium Equation for Bonds

• When the short-term interest rate is the only source of uncertainty, the following partial differential equation must be satisfied by any zero-coupon bond (see equation (24.18) in the textbook)

\[
\frac{1}{2} \sigma(r)^2 \frac{\partial^2 P}{\partial r^2} + [\alpha(r) - \sigma(r) \phi(r, t)] \frac{\partial P}{\partial r} + \frac{\partial P}{\partial t} - rP = 0
\]

where

1. \( r \) denotes the short-term interest rate, which follows the Ito process

\[ dr = a(r) dt + \sigma(r) dZ; \]

2. \( \phi(r, t) \) is the Sharpe ratio corresponding to the source of uncertainty \( Z \), i.e.,

\[ \phi(r, t) = \frac{\alpha(r, t, T) - r}{q(r, t, T)} \]

with the coefficients \( P \cdot \alpha \) and \( P \cdot q \) are the drift and the volatility (respectively) of the Ito process \( P \) which represents the bond-price for the interest-rate \( r \).

• This equation characterizes claims that are a function of the interest rate (as there are no alternative sources of uncertainty).
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The risk-neutral process for the interest rate

- The risk-neutral process for the interest rate is obtained by subtracting the risk premium from the drift:
  \[ dr_t = [a(r_t) - \sigma(r_t)\phi(r_t, t)] dt + \sigma(r_t) dZ_t \]

- Given a zero-coupon bond, Cox et al. (1985) show that the solution to the equilibrium equation for the zero-coupon bonds must be of the form (see equation (24.20) in the textbook)
  \[ P[t, T, r(t)] = \mathbb{E}^*[e^{-R(t, T)}] \]

where

1. \( \mathbb{E}^*_t \) represents the expectation taken with respect to risk-neutral probabilities given that we know the past up to time \( t \);
2. \( R(t, T) \) represents the cumulative interest rate over time, i.e., it satisfies the equation (see (24.21) in the book)
  \[ R(t, T) = \int_t^T r(s) \, ds \]

- Thus, to value a zero-coupon bond, we take the expectation over “all the discount factors” implied by these paths.
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Summary

- One approach to modeling bond prices is exactly the same procedure used to price options on stock.
- We begin with a model of the interest rate and then use Ito’s Lemma to obtain a partial differential equation that describes the bond price - the equilibrium equation.
- Next, using the PDE together with boundary conditions, we can determine the price of the bond.
- In the present course, we skip the details - you will simply use the formulae that are the end-product of this strategy.
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