NAME:

M339W/389W Financial Mathematics for Actuarial Applications
University of Texas at Austin
In-Term Exam II
Instructor: Milica Čudina

Notes: This is a closed book and closed notes exam. The maximum number of points on this exam is 75.

Time: 50 minutes

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MULTIPLE CHOICE

| 2.9 (5) | a | b | c | d | e |
| 2.10 (5)| a | b | c | d | e |
| 2.11 (5)| a | b | c | d | e |
| 2.12 (5)| a | b | c | d | e |
| 2.13 (5)| a | b | c | d | e |
| 2.14 (5)| a | b | c | d | e |

FOR GRADER'S USE ONLY:

<table>
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<tr>
<th>DEF’N</th>
<th>T/F</th>
<th>2.6</th>
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<th>Σ</th>
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2.1. **TRUE/FALSE QUESTIONS.** Please note your answers on the front page.

**Problem 2.1.** (2 points) Under the risk-neutral probability measure, every option on a particular stock has the continuously compounded, risk-free interest rate as its mean rate of return. *True or false?*

**Solution:** TRUE

**Problem 2.2.** Assume the Black-Scholes stock-pricing model is in force. Let \( E^* \) denote the expectation under the risk-neutral probability measure \( P^* \). Let \( \{S(t), t \geq 0\} \) denote the price of a continuous-dividend-paying stock. Then, in our usual notation,

\[
E^*[S(T)] = S(0)e^{(r-\delta)T}.
\]

*True or false?*

**Solution:** TRUE

**Problem 2.3.** The Black-Scholes option pricing formula can **always** be used for pricing American-type call options on non-dividend-paying assets. *True or false?*

**Solution:** TRUE

**Problem 2.4.** Let the stock price be modeled by a lognormal distribution. Then, the median stock price always exceeds the mean stock price. *True or false?*

**Solution:** FALSE

**Problem 2.5.** Let the stock price be modeled by a lognormal distribution. Then, the expected payoff of a European put option with exercise date \( T \) and strike \( K \) greater than or equal to \( \max(0, K - \mathbb{E}[S(T)]) \). *True or false?*

**Solution:** TRUE
2.2. **FREE-RESPONSE PROBLEMS.** Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they’re correct) are worth 0 points.

**Problem 2.6.** (10 points)
Consider a non-dividend-paying stock whose price is modeled using the lognormal distribution. Suppose that the current stock price equals $100 and that its volatility is given to be 0.2.

The continuously compounded, risk-free interest rate is assumed to equal 0.04.

Consider a derivative security which entitles its owner to obtain a European call option on the above stock six months from today, i.e., at time $t^* = 1/2$. The call option is to be half a year to expiration at time of delivery and have the strike equal to 105% of the time–$t^*$ price of the underlying asset. This contract is called a **forward start option**.

What is the price of the forward start option?

**Solution:**

At time $t^* = 1/2$, the Black-Scholes price of the call option to be delivered equals

$$V_C(t^*) = S(t^*)N(d_1) - 1.05S(t^*)e^{-r(T-t^*)}N(d_2)$$

$$= S(t^*)(N(d_1) - 1.05e^{-0.02}N(d_2))$$

with

$$d_1 = \frac{1}{0.2\sqrt{0.5}} \left[ -\ln(1.05) + \left( 0.04 + \frac{0.2^2}{2} \right) \times \frac{1}{2} \right] = -0.13,$$

$$d_2 = d_1 - \sigma \sqrt{T-t^*} = -0.27.$$  

Hence,

$$V_C(t^*) = S(t^*)(0.4483 - 1.05e^{-0.02} \times 0.3936) = S(t^*)(0.0432).$$

So, one would need to buy 0.0432 shares of stock to be able to buy the call option in question at time–$t^*$. This amount of shares costs $4.32.

The price of the forward start option is
Problem 2.7. (10 points) You observe the following stock prices in the beginning of every month:

<table>
<thead>
<tr>
<th>Observation</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock price</td>
<td>100</td>
<td>90</td>
<td>90</td>
<td>81</td>
<td>90</td>
<td>100</td>
<td>90</td>
<td>81</td>
<td>90</td>
<td>100</td>
<td>100</td>
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i. (5 points) Calculate the (unbiased) estimate of the annual volatility based on the above data.

ii. (5 points) Assuming that the stock-prices are modeled using a geometric Brownian motion, what is your estimate of the stock’s rate of appreciation?

Solution:

i. The initial and the final stock prices are both 100, so the observed average of the log-ratios is 0. The log-ratios based on the above data are

<table>
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<tr>
<th>$r_1$</th>
<th>$r_2$</th>
<th>$r_3$</th>
<th>$r_4$</th>
<th>$r_5$</th>
<th>$r_6$</th>
<th>$r_7$</th>
<th>$r_8$</th>
<th>$r_9$</th>
<th>$r_{10}$</th>
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<td>$\ln \left( \frac{9}{10} \right)$</td>
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So, the unbiased estimate of the volatility squared (on the monthly scale) is

$$\frac{1}{10 - 1} \times 8 \times \left( \ln \left( \frac{9}{10} \right) \right)^2$$

The annualized squared volatility estimate is

$$\hat{\sigma}^2 = 12 \times \frac{8}{9} \left( \ln \left( \frac{10}{9} \right) \right)^2.$$  

Finally, $\hat{\sigma} = 0.344106$.

ii. Recall that the observed average of the log-ratios of stock prices equals zero. So,

$$\hat{\alpha} - \hat{\delta} = \frac{1}{2} \hat{\sigma}^2 = 0.0592045.$$
Problem 2.8. (15 points)

You are considering an investment in a non-dividend-paying stock versus an investment in a savings account. According to your belief, the stock’s mean rate of return is $\alpha$ and its volatility is $\sigma$.

The continuously compounded interest rate is equal to $r$.

What is the probability that the stock outperforms the savings account at time $-T$? You should leave your final answer in terms of the function $N$.

Solution: With $Z \sim N(0,1)$, we are looking for the probability

$$\mathbb{P}[S(T) > S(0)e^{rT}] = \mathbb{P}\left[ S(0)e^{(\alpha-\sigma^2/2)T + \sigma\sqrt{T}Z} > S(0)e^{rT} \right]$$

$$= \mathbb{P}\left[ (\alpha - \sigma^2/2)T + \sigma\sqrt{T}Z > rT \right]$$

$$= \mathbb{P}\left[ Z > \frac{\sqrt{T}}{\sigma}(r - \alpha + \sigma^2/2) \right]$$

$$= N\left( \frac{(\alpha - r - \sigma^2/2)\sqrt{T}}{\sigma} \right).$$
2.3. **MULTIPLE CHOICE QUESTIONS.** Please note your answers on the front page.

**Problem 2.9.** Assume the Black-Scholes setting. The stock price today is $60.00 per share, its dividend yield is \(0.01\) and its volatility is \(0.20\).

The continuously-compounded, risk-free interest rate is \(0.04\).

Alice wagers to pay $100 to Bob if the price of the above stock in 1 year is above $75.00. What is the "fair" price of this wager?

(a) 13.77
(b) 14.67
(c) 15.57
(d) 16.46
(e) None of the above.

**Solution:** (a)

This is the Black-Scholes price of 100 **cash calls**, i.e.,

\[
100V_{CC}(0) = 100e^{-0.04} N(d_2)
\]

with

\[
d_2 = \frac{1}{0.2} \left( \ln\left(\frac{60}{75}\right) + (0.04 - 0.01 - \frac{1}{2} \cdot 0.2^2) \right) = 5(\ln(0.8) + 0.01) = -1.07
\]

So,

\[
100V_{CC}(0) = 100e^{-0.04}(0.143276) = 13.77.
\]

**Problem 2.10.** Assume the Black-Scholes setting.

Today’s price of a non-dividend paying stock is $65, and its volatility is \(0.20\).

The continuously-compounded, risk-free interest rate is \(0.055\).

What is the price of a three-month, $60-strike European put option on the above stock?

(a) 0.66
(b) 0.59
(c) 0.44
(d) 0.37
(e) None of the above.

**Solution:** (b)

In our usual notation, the price is

\[
V_P(0) = Ke^{-rT}N(-d_2) - S(0)N(-d_1)
\]

with

\[
d_1 = \frac{1}{0.2\sqrt{1/4}} \left( \ln\left(\frac{65}{60}\right) + (0.055 + \frac{1}{2} \cdot 0.2^2) \left(\frac{1}{4}\right) \right) = 10(\ln(65/60) + (0.075)(0.25)) = 0.99,
\]

\[
d_2 = d_1 - 0.2\sqrt{0.25} = 0.89.
\]
So,

\[ V_P(0) = 60e^{-0.055 \cdot \frac{1}{2}} (1 - 0.8133) - 65 \cdot (1 - 0.8389) = 0.5922. \]

**Problem 2.11.** Assume the Black-Scholes setting.

Today’s stock price is observed to be \( S(0) = $30 \) per share. Its dividend yield is given to be 0.01 and its volatility equals 0.30.

The continuously-compounded, risk-free interest rate is \( r = 0.04 \).

What is the price of a half-year, $30-strike put?

- (a) 2.75
- (b) 2.38
- (c) 1.80
- (d) 1.20
- (e) None of the above.

**Solution: (b)**

In our usual notation, the price is

\[ V_P(0) = Ke^{-rT} N(-d_2) - S(0)e^{-\delta T} N(-d_1) \]

with

\[ d_1 = \frac{1}{0.30 \sqrt{0.5}} \left[ \ln(30/30) + (0.04 - 0.01 + 0.045)(0.5) \right] = \frac{0.075}{0.3} \sqrt{0.5} = 0.18, \]

\[ d_2 = d_1 - 0.3 \sqrt{0.5} = -0.04. \]

So, our final answer is

\[ V_P(0) = Ke^{-rT} N(-d_2) - S(0)e^{-\delta T} N(-d_1) \]
\[ = 30 \left[ e^{-0.04/2} (0.516) - e^{-0.01/2} (1 - 0.5714) \right] \]
\[ = 2.3796. \]
**Problem 2.12.** The current price of a continuous-dividend paying stock is observed to be $50 per share while its volatility is given to be 0.34. The dividend yield is projected to be 0.02. The continuously compounded, risk-free interest rate is 0.05.

Consider a European call option with the strike price equal to $40 and the exercise date in three months.

Using the Black-Scholes pricing formula, find the value $V_C(0)$ of this option at time $t = 0$.

(a) $9.08$
(b) $9.80$
(c) $10.55$
(d) $14.10$
(e) None of the above.

**Solution:** (c)

In our usual notation,

$$d_1 = \frac{1}{\sigma \sqrt{T}} \left[ \ln \left( \frac{S(0)}{K} \right) + \left( r - \delta + \frac{1}{2} \sigma^2 \right) T \right]$$

$$= \frac{1}{0.34 \sqrt{1/4}} \left[ \ln \left( \frac{50}{40} \right) + \left( 0.05 - 0.02 + \frac{1}{2} \times 0.34^2 \right) \times \frac{1}{4} \right] = 1.44,$$

$$d_2 = d_1 - \sigma \sqrt{T} = 1.27.$$

The standard normal tables give us

$$N(d_1) = 0.9253, \quad N(d_2) = 0.8983.$$

Finally,

$$V_C(0) = S(0)e^{-\delta T}N(d_1) - Ke^{-rT}N(d_2) = 10.55.$$

**Problem 2.13.** Assume the lognormal stock-price model. The current stock price is $140 per share. Its rate of appreciation is 0.12 and its volatility is 0.32.

Your random-number generator gives you the following three draws from the unit uniform distribution:

$$1 - 0.5478, \quad 0.8365, \quad 0.5359.$$

What is the average of the three time-2 stock prices you obtain using the above three draws?

(a) $160.65$
(b) $177.98$
(c) $189.94$
(d) $246.94$
(e) None of the above.

**Solution:** (c)

According to the lognormal stock-price model, we have

$$S(2) = S(0)e^{(\alpha - \delta - \sigma^2/2)(2) + \sigma \sqrt{2Z}}$$
with $$Z \sim N(0, 1)$$. So, our three simulated stock-price values are

$$s_1 = S(0)e^{(\alpha - \delta - \sigma^2/2)(2) + \sigma \sqrt{2} N^{-1}(-0.5478)} = 152.16,$$

$$s_2 = S(0)e^{(\alpha - \delta - \sigma^2/2)(2) + \sigma \sqrt{2} N^{-1}(0.8365)} = 250.32,$$

$$s_3 = S(0)e^{(\alpha - \delta - \sigma^2/2)(2) + \sigma \sqrt{2} N^{-1}(0.5359)} = 167.33.$$  

So, our answer is

$$\frac{1}{3}(152.16 + 250.32 + 167.33) = 189.94.$$  

**Problem 2.14.** Assume the Black-Scholes framework. You are given the following information for a stock that pays dividends continuously at a rate proportional to its price:

(i) The current stock price is $250.
(ii) The stocks volatility is 0.3.
(iii) The continuously compounded expected rate of stock-price appreciation is 15%.

Find the value $$s^*$$ such that

$$\mathbb{P}[S(4) > s^*] = 0.05.$$  

(a) $861.65
(b) $874.18
(c) $889.94
(d) $905.48
(e) None of the above.

**Solution:** (e)

$$s^* = 250e^{(0.15 - 0.045)(4) + 0.3(2)(1.645)} = 1020.92$$