Provide a complete solution to the following problem(s):

**Problem 6.1.** Let $Z$ be a standard Brownian motion. Define the process $Y = \{Y(t), t \geq 0\}$ as $Y(t) = e^{t/2} \sin(Z(t))$. We will show that this process is a martingale.

a. (5 points) Find the expression for $d(\sin(Z(t)))$ using Itô’s Lemma.

b. (5 points) Use your answer to part a. to find the differential representation $dY(t)$ of the process $Y$.

c. (2 points) Now, it is sufficient to prove that the drift term in the representation obtained in part b. is equal to zero. Verify this.

**Problem 6.2.** (14 points) Solve problem 20.3 from McDonald.

**Problem 6.3.** (2 points) In the setting of the Black-Scholes stock-price model, let $\{S(t), t \geq 0\}$ denote the stock price. Define the new stochastic process $X(t) = \ln(S(t))$, for every $t \geq 0$.

Then we have that the stochastic process $\{X(t), t \geq 0\}$ is a geometric Brownian motion. True or false?

**Problem 6.4.** (2 points) Let $\{Z(t), t \geq 0\}$ denote a standard Brownian motion. Then the stochastic process $\{U(t), t \geq 0\}$ defined as

$$U(t) = Z(t)^2 - 2t,$$

for every $t \geq 0$ has zero drift. True or false?

**Problem 6.5.** (2 points) In the setting of the Black-Scholes stock-price model, let $\{S(t), t \geq 0\}$ denote the stock price with parameters $\alpha$ and volatility $\sigma$. Define the new stochastic process $X(t) = \ln(S(t))$, for every $t \geq 0$.

Then we have that

$$\text{Var}[X(t+h) - X(t)] = \sigma^2 h,$$

for every $t \geq 0$ and $h > 0$.

True or false?

**Problem 6.6.** (2 points) In the setting of the Black-Scholes stock-price model, let $\{S(t), t \geq 0\}$ denote the stock price with volatility $\sigma$ and drift $\alpha$. Then, we have that

$$\text{Var}[S(t+h) \mid S(t)] \approx S(t)^2 \sigma^2 h,$$

for every $t \geq 0$ and infinitesimally small $h > 0$.

True or false?

**Problem 6.7.** (10 points) Use Itô’s Lemma to express $dF(S(t))$ for $F : \mathbb{R}_+ \to \mathbb{R}_+$ given as $F(x) = \sqrt{x}$, where the stochastic process $\{S(t), t \geq 0\}$ satisfies the stochastic differential equation

$$dS(t) = a(b - S(t)) dt + \sigma \sqrt{S(t)} dZ(t)$$

with $a, b$ and $\sigma$ positive constants and $\{Z(t), t \geq 0\}$ a standard Brownian motion.

**Problem 6.8.** (2 points) Let $\{Z(t), t \geq 0\}$ be a standard Brownian motion. Then the process

$$V(t) = t^2 Z(t) - 2 \int_0^t sZ(s) ds$$

has zero drift. True or false?