Formulas.

If $X$ has the binomial distribution with parameters $n$ and $p$, then $\mathbb{P}[X = k] = \binom{n}{k} p^k (1 - p)^{n-k}$, for $k = 0, \ldots, n$, $\mathbb{E}[X] = np$, $\text{Var}[X] = np(1 - p)$.

If $X$ has the standard normal distribution, then its mean is zero, its variance is one, and its density equals

$$
\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad x \in \mathbb{R}.
$$

True/false questions.

Problem 1.1. (2 points) The following statement makes sense:

The correlation between eye color and height equals 0.05.

True or false?

Solution: FALSE

Problem 1.2. (2 points)

We know that the distribution of wealth in the United States is right skewed. Thus, the majority of people in the US have more than the average (i.e. mean) level of wealth. True or false?

Solution: FALSE

Since the wealth distribution is skewed to the right, the mean is larger than the median. Thus, more than half people have less than the average level of wealth.

Free-response problems.

Problem 1.3. (15 points)

The Alarming Studies Weekly reported on a study suggesting that

"frequently 'heading' the ball in soccer lowers players' IQs".

A psychologist tested 60 male soccer players, ages 14 – 29, who played soccer up to 5 times a week. Players who averaged 10 or more “headers” a game had an average IQ of 103, while players who “headed” 1 or fewer times per game had an average IQ of 112.

(a) (2 points) Describe the population of interest to the psychologist.

Solution: Soccer players.

(b) (2 points) Describe the sample.

Solution: Quote: "\ldots 60 male soccer players, ages 14 – 29, who played up to 5 times a week \ldots \"

(c) (4 points) Identify the variables of interest, i.e., the explanatory and response variables.

Solution: Average number of "headers" per game (explanatory) and IQ (response).

(d) (2 points) Identify the type of the variables in (c). Are they categorical or quantitative?

Solution: Both quantitative.
(e) (5 points)
What do you think the inference (i.e. conclusions) made by the psychologist is? Discuss in short (two or
three sentences) the possible reasons why this inference might be misleading!

Solution:
The conclusion is, of course: 
"frequently “heading” the ball in soccer lowers players’ IQs".

The remarks are multiple: from one point of view the sample is not random enough: only male players
of certain age that play soccer in a semi-professional manner. On the other hand, the sample is also too
heterogenous including both people that play soccer once or twice a week (therefore less experienced and
"head" the ball less!), as well as the players that play soccer 5 times a week (and are in no way reluctant to
"head" the ball!). Those two groups might have been a cause of discrepancy in the results.

These are just some possibilities, any reasonable remark was good enough!

Problem 1.4. (12 points)
The population of Midsomer University students can roughly be divided into two subgroups - coffee addicts and
tea addicts. GPAs of both populations are approximately normally distributed - \(N(mean = 3.6, sd = 0.25)\) for coffee
addicts and \(N(mean = 3.4, sd = 0.1)\) for tea addicts.

(a) (5 points) What is the probability that a coffee addict chosen at random has a GPA higher than 3.7?

Solution: For \(x = 3.7\), the \(z\)-score is
\[
z = \frac{3.7 - 3.6}{0.25} = 0.4,
\]
so the probability is approximately 1-0.655=0.34=34% .

(b) (5 points) What is the range of GPAs for the top 15% of the tea addicts?

Solution: The \(z\)-score corresponding to \(1 - 15\% = 85\%\) is 1.04. So, we solve for \(x\) in
\[
1.04 = \frac{x - 3.4}{0.1},
\]
and get \(x = 3.4 + 0.104 = 3.504\). Therefore the range of GPAs for the top 15% of the tea-drinkers is [3.504, 4.0].

(c) (2 points) Would it be correct to say that the GPAs of the population of all Midsomer University students
follow the normal distribution (given the assumptions of the problem)?

Solution: No, the resulting population will be very inhomogeneous, having two modes (peaks). (Remember
the example of men’s and women’s heights put together.)

Problem 1.5. (17 points)
Suppose that the thumb sizes of the US males follow a normal distribution with an unknown mean \(\mu\) and standard
deviation \(\sigma = 20\) on the GPI - scale (Grey’s Pollex Index - GPI - from 50 to 280). The US Department of Thumbs
and Toes (DTT) reports that the mean thumb size in the country is \(\mu = 150\).

Being the chairman of the Faculty of Thumbs of the local university you see an excellent opportunity here and
decide to conduct your own study of the size of the average American thumb. After collecting a SRS of 100 American
thumbs you obtain the following sample average \(\bar{x} = 153\).

1 (5 pts) Construct a 95%-confidence interval for the unknown parameter \(\mu\) based on your study.

Solution:
\[
\bar{x} \pm z^* \frac{\sigma}{\sqrt{n}} \text{ i.e. } 153 \pm 3.92 \text{ i.e. } (149.08, 156.92).
\]
ii. (7 pts)
Assess the strength of evidence your study carries against the DTT findings. In other words: state the hypotheses, carry out the hypothesis test, and report your findings.

Solution:
The hypotheses are
\[
\begin{align*}
H_0 : \mu &= 150 \\
H_a : \mu &> 150
\end{align*}
\]
To get the p-value we calculate $P[\bar{X} > 153]$. A simple z-score calculation gives us that the p-value is 6.68%.

(iii) (5 pts) You dream of achieving fame and fortune by being the first person ever to estimate the mean thumb size up to the margin of error equal to ±0.1. How large a sample size do you need for that?

Solution:
\[
n = \left( \frac{z \cdot \sigma}{m} \right)^2 = 400^2 = 160000.
\]

Problem 1.6. Don’t mess with Texas!

The Anti-Littering League wishes to gauge the success of the ingenious Don’t mess with Texas! campaign.

Realizing the obvious problems with conducting a survey which outright asks the questions: “Are you or have you ever been a litterer?”, they resort to the randomized-response method.

They prompt a computer to display the question

“Have you ever littered?”

with probability 0.6. The rest of the time, a virtual fair coin is flipped on the screen and the subject is asked

“Is the outcome heads or tails?”

In both cases, the subject is prompted to click the button with Yes or No. The interviewer did not know the actual question asked, just the ultimate response. So, there was no real reason for the subject to lie, and we assume that the subjects responded truthfully.

i. (5 points)
It turned out that 50% of the subjects answered “yes”. Give an estimate of the proportion of litterers in this population.

Solution: Now, we are given that $P[D] = 0.50$ with the event $D$ defined as

$D = \{\text{the subject answered Yes}\}$.

Our goal is to figure out $P[D \mid C]$ with the conditioning event $C$ given by

$C = \{\text{the subject was asked the littering question}\}$.

By the Law of Total Probability,

$P[D] = P[D \cap C] + P[D \cap C^c] = P[D \mid C]P[C] + P[D \mid C^c]P[C^c]$.

So,

$0.6P[D \mid C] = 0.50 - 0.5 \times 0.4 = 0.3 \Rightarrow P[D \mid C] = 0.5$.

ii. (5 points) What percentage of “yes” answers would you have obtained in an ideal world in which nobody ever litters?
\[ P[D] = P[D \cap C] + P[D \cap C^c] \\
= P[D \mid C]P[C] + P[D \mid C^c]P[C^c] \\
= 0 \times 0.6 + 0.5 \times 0.4 = 0.2. \]

Multiple-choice problems.

Problem 1.7. (5 points) A medical researcher thinks that adding calcium to the diet will help reduce blood pressure. She believes that the effect is different for men and women. 20 men and 20 women are willing to participate in the study. The researcher chooses 10 of the men and 10 of the women at random. These chosen 20 men and women take a calcium pill every day. The other 20 men and women take a placebo. This is a . . .

a.: stratified random sample design.

b.: simple random sample design.

c.: randomized block experimental design.

d.: completely randomized experimental design.

e.: None of the above is correct.

Solution: c.

Problem 1.8. (5 points)

In a hypothesis testing problem, p-value = 3% means that . . .

a.: Null hypothesis has a 3% chance to be wrong.

b.: If the null hypothesis is true, the probability of observing as extreme or more extreme than what was observed is 3%.

c.: Alternative hypothesis has a 3% chance to be wrong.

d.: If we repeat the procedure a lot times, approximately 3% of the tests will be significant.

e.: None of the above.

Solution: b.

Problem 1.9. (5 points)

A new headache remedy is given to a group of 250 patients who suffer severe headaches. Of these patients, 200 report that the remedy is very helpful in treating their headaches. From this information you conclude

a.: The remedy is effective for the treatment of headaches.

b.: Nothing, because the sample size is too small.

c.: The new treatment is better than aspirin.

d.: Nothing, because there is no control group.

e.: None of the above.

Solution: d.

Problem 1.10. (5 points)

To estimate a population mean, our resident statistician Martyn Rivera plans to pick two simple random samples, each of size 100, from the population. He also plans to calculate the confidence interval with level \( C \) for each sample. What is the probability that at least one of his confidence intervals will cover the population mean?
a.: $C^2$
b.: $1 - C^2$
c.: $2C$
d.: $1 - (1 - C)^2$
e.: None of the above

Solution: d.