An interpolation result for the convergence to the Hartree dynamics in Sobolev trace norms

Michael Hott

University of Texas at Austin

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Outline

Introduction
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Previous results

Our results

Proof of both Theorems
Physical background

Nernst 1906:
If one cools a system down, it will drop in to the lowest quantum state (Third law of TD).
For a Bose gas, we may compute the density in the limit of a large system as

$$\frac{N}{V} = c T^3$$

\[\int_0^\infty \sqrt{z} dz e^{z - \beta \mu - 1}\]

where \(\beta = \left(\frac{k_B}{T}\right)^{-1}\) and \(z = \beta \epsilon\).

For fixed \(\frac{N}{V}\), as \(T\) decreases, \(\mu\) increases and eventually \(\mu(T_c) = 0\) for some critical temperature \(T_c\).
Below \(T_c\) we would have \(\mu > 0\) for a Bose gas, a contradiction!

→ Resolve paradox by taking \(\epsilon = 0\) states separately into account, i.e., below \(T_c\) these states become relevant!

→ Bose-Einstein Condensation
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Setting

$N$ weakly interacting bosons

$$H_N := \sum_{i=1}^{N} T_i + \frac{1}{N-1} \sum_{i<j} \frac{\lambda}{|x_i - x_j|} \quad \text{on} \quad \mathcal{H}_N := L^2(\mathbb{R}^3) \otimes s^N$$
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$\lambda \in \mathbb{R}$ some coupling constant. In this talk, focus on $T = -\Delta$. 

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Assume a BEC in the beginning. Evolution equation:

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\begin{aligned}
    i \partial_t \Psi_{N,t} &= H_N \Psi_{N,t} \\
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\end{aligned}
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(1)
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Solution: $\Psi_{N,t} = e^{-iH_N t} \Psi_{N,0}$.

Problem: We are interested in measurable quantities like $\langle \Psi_{N,t}, A \Psi_{N,t} \rangle$. 

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Problem: We are interested in measurable quantities like $\langle \Psi_{N,t}, A \Psi_{N,t} \rangle$. \(\rightarrow\) In general hard to compute!
Effective equation for (1)

**Ansatz:** $\Psi_{N,t} = \phi_t \otimes N$

Insert in (1):

$$N \langle \phi_t, i \partial_t \phi_t \rangle = \langle \Psi_{N,t}, H_N \Psi_{N,t} \rangle = N \langle \phi_t, T \phi_t \rangle + N^2 \langle \phi_t \otimes 2, \lambda |x_1 - x_2| \phi_t \otimes 2 \rangle =: N h(\phi_t, \phi_t)$$

Obtain dynamics:

$$i \partial_t \phi_t = \partial_t \phi_t h(\phi_t, \phi_t) = T \phi_t + \lambda |\cdot| \star |\cdot| \phi_t \big|_t = 0 = \phi_0.$$
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    i \partial_t \varphi_t = \partial_{\bar{\varphi}_t} h(\varphi_t, \bar{\varphi}_t) = T \varphi_t + \frac{\lambda}{|\cdot|} \ast |\varphi_t|^2 \varphi_t \\
    \varphi_t \bigg|_{t=0} = \varphi_0
\end{cases}$$

(2)
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- Many results known for the Hartree equation

\[ E_H(\varphi_t) := \|\nabla \varphi_t\|_2^2 + \langle \varphi_t, \frac{\lambda}{|.|} \ast |\varphi_t|^2 \varphi_t \rangle \] is conserved! \( \Rightarrow \) (2) is globally well-posed for \( H^1 \)-initial data
In what sense is (2) an effective equation for (1)?
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First result but for bounded potentials instead of the Coulomb potential was obtained by [Sp80]:

\[
\langle \psi_{N,t}, A \otimes I_{N-k} \psi_{N,t} \rangle = \langle \phi_t^{\otimes k}, A \phi_t^{\otimes k} \rangle + o(1) \quad (3)
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as \( N \to \infty \) for \( A \in \mathcal{B}(L^2(\mathbb{R}^3)^{\otimes s^k}), 1 \leq k \leq N \).
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MS12 : $T = \sqrt{1 - \Delta}$, $O(N^{-\frac{1}{4}})$ for unbounded $A$!
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**Lu12** : $T = (i\nabla - A)^2, O(N^{-\frac{1}{4}})$ for unbounded $\mathcal{A}$.
How does one usually prove (3)?
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We have

\[
\left| \langle \psi_{N,t}, A \otimes I_{N-k} \psi_{N,t} \rangle - \langle \varphi^{\otimes k}_t, A \varphi^{\otimes k}_t \rangle \right|
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\]

\[
= \left| \text{Tr} \left( \mathcal{A} \right) \right| \left( \text{Tr}_{k+1, \ldots, N} \langle \psi_{N,t} \rangle \langle \psi_{N,t}^{\dagger} \rangle \right) =: \gamma_{N,t}^{(k)}
\]

\[
- \text{Tr} \left( \mathcal{A} \varphi_{t}^{\otimes k} \right) \left( \varphi_{t}^{\otimes k} \right) \left( \varphi_{t}^{\otimes k} \right) \left( \varphi_{t}^{\otimes k} \right) =: P_{t}^{(k)}
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\[ = |\text{Tr}(A \underbrace{\text{Tr}_{k+1,\ldots,N} \Psi_{N,t} \langle \Psi_{N,t} |}_{=: \gamma_{N,t}^{(k)}}) | \]

\[ - \text{Tr}(A \underbrace{\langle \varphi_{t}^{\otimes k} | \langle \varphi_{t}^{\otimes k} |}_{=: P_{t}^{(k)}}) | \]

\[ \leq \| A \| \| \text{Tr} \| \gamma_{N,t}^{(k)} - P_{t}^{(k)} \| \]
How does one usually prove (3)?

We have

\[ \left| \langle \psi_{N,t}, A \otimes I_{N-k} \psi_{N,t} \rangle - \langle \varphi_t^k, A \varphi_t^k \rangle \right| = \left| \text{Tr} \left( S_k^{-\theta/2} A S_k^{-\theta/2} S_k^{\theta/2} \left[ \text{Tr}_{k+1,\ldots,N} \psi_{N,t} \langle \psi_{N,t}, S_k^{\theta/2} \rangle \right] \right) \right| =: \gamma_{N,t}^{(k)} \]

\[ - \text{Tr} \left( S_k^{-\theta/2} A S_k^{-\theta/2} S_k^{\theta/2} \left[ \varphi_t^k \langle \varphi_t^k, S_k^{\theta/2} \rangle \right] \right) =: P_t^{(k)} \]

\[ \leq \left\| S_k^{-\theta/2} A S_k^{-\theta/2} \right\| \text{Tr} \left( S_k^{\theta/2} \left( \gamma_{N,t}^{(k)} - P_t^{(k)} \right) S_k^{\theta/2} \right), \]

where \( S_k := \sum_{i=1}^{k} (1 + T_i) \).
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We have

$$\left| \langle \Psi_{N,t}, A \otimes I_N - k \Psi_{N,t} \rangle - \langle \varphi \otimes k, A \varphi \otimes k \rangle \right|$$

$$= |\text{Tr}(S_k^{-\theta/2} A S_k^{-\theta/2} S_k^{\theta/2} \underbrace{\text{Tr}_{k+1,\ldots,N} \langle \Psi_{N,t}, \Psi_{N,t} | S_k^{\theta/2} \rangle}_{=: \gamma_{N,t}^{(k)}}) - \text{Tr}(S_k^{-\theta/2} A S_k^{-\theta/2} S_k^{\theta/2} \langle \varphi \otimes k, \varphi \otimes k | S_k^{\theta/2} \rangle)|$$

$$=: P_t^{(k)}$$

$$\leq \|S_k^{-\theta/2} A S_k^{-\theta/2}\| \text{Tr} |S_k^{\theta/2}(\gamma_{N,t}^{(k)} - P_t^{(k)}) S_k^{\theta/2}|,$$

where $S_k := \sum_{i=1}^{k} (1 + T_i)$. Define also the Pickl functional $a_{N,t} := \langle \Psi_{N,t}, (1 - (|\varphi_t \rangle \langle \varphi_t|)_1) \Psi_{N,t} \rangle$, see [Pi11].
Main Theorem

Theorem (Anapolitanos, Hott 2016)
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Assume $T = -\Delta$ and $\varphi_0 \in H^1(\mathbb{R}^3)$. 

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Main Theorem

Theorem (Anapolitanos, Hott 2016)

Assume $T = -\Delta$ and $\varphi_0 \in H^1(\mathbb{R}^3)$.

(i) For any $\theta \in [0, 1)$ there exists a constant $C > 0$ such that for any $k \in \mathbb{N}$, $N \geq k$ and any $t > 0$ we have

$$\text{Tr} \left| S_k^{\theta/2} (\gamma_{N,t}^{(k)} - P_t^{(k)}) S_k^{\theta/2} \right| \leq Ck(a_{N,t}^{\min(\frac{1}{2}, 1-\theta)} + \|\gamma_{N,t}^{(k)} - P_t^{(k)}\|_{HS}^{1-\theta}).$$
Main Theorem

Theorem (Anapolitanos, Hott 2016)

Assume $T = -\Delta$ and $\varphi_0 \in H^1(\mathbb{R}^3)$.

(i) For any $\theta \in [0, 1)$ there exists a constant $C > 0$ such that for any $k \in \mathbb{N}$, $N \geq k$ and any $t > 0$ we have

$$\text{Tr} \left| S_k^\theta \left( \gamma_{N,t}^{(k)} - P_t^{(k)} \right) S_k^\theta \right| \leq C k \left( a_{N,t}^\min(\frac{1}{2}, 1-\theta) + \| \gamma_{N,t}^{(k)} - P_t^{(k)} \|_{HS}^{1-\theta} \right).$$

(ii) For all $k \in \mathbb{N}$ and $t > 0$ we have

$$\lim_{N \to \infty} \text{Tr} \left| S_k^{\frac{1}{2}} \left( \gamma_{N,t}^{(k)} - P_t^{(k)} \right) S_k^{\frac{1}{2}} \right| = 0.$$
How can one apply this theorem?
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Theorem (Pickl 2011; Knowles, Pickl 2010)

Under the assumptions of the main theorem there exist constants $C, D \in \mathbb{R}$ independent of $N, t$ such that for any $N \in \mathbb{N}$ and $t > 0$ we have $a_{N, t} \leq \frac{Ce^{Dt}}{N}$. 
How can one apply this theorem?

**Theorem (Pickl 2011; Knowles, Pickl 2010)**

*Under the assumptions of the main theorem there exist constants $C, D \in \mathbb{R}$ independent of $N, t$ such that for any $N \in \mathbb{N}$ and $t > 0$ we have $a_{N,t} \leq \frac{Ce^{Dt}}{N}$.*

**Theorem (Chen, Lee, Schlein 2011)**

*We assume again the conditions of the main theorem. Then for any $k \in \mathbb{N}$ there exist $C_k, D_k \in \mathbb{R}$ such that for any $N \in \mathbb{N}$ with $N \geq k$ and any $t > 0$ we have $\|\gamma_{N,t}^{(k)} - P_t^{(k)}\|_{HS} \leq \frac{C_k e^{D_k t}}{N}$.*
Corollary

Assume $T = -\Delta$ and $\phi_0 \in H^1(R^3)$. Let $k \in \mathbb{N}$ and $A$ be a self-adjoint operator acting on $L^2(R^3) \otimes S^k$. Assume that there exists $\theta \in [0, 1)$ such that $S^{-\theta/2}kA S^{-\theta/2}k$ can be extended to a bounded operator on $L^2(R^3) \otimes S^k$ with operator norm $\|S^{-\theta/2}kA S^{-\theta/2}k\|$. 

(i) If $\theta < 1$, there exist $C_k, D_k > 0$ independent of $N$, $t$ such that for any $N \in \mathbb{N}$ with $N \geq k$ and any $t > 0$ we have

\[
\left| \langle \Psi_N, t, A \otimes I_N - k \Psi_N, t \rangle - \langle \phi \otimes k t, A \phi \otimes k t \rangle \right| \leq C_k e^{D_k t N \min\left(\frac{1}{2}, 1 - \theta\right)} \|S^{-\theta/2}kA S^{-\theta/2}k\|.
\]

(ii) If $\theta = 1$, then

\[
\lim_{N \to \infty} |\langle \Psi_N, t, A \otimes I_N - k \Psi_N, t \rangle - \langle \phi \otimes k t, A \phi \otimes k t \rangle| = 0 \quad \forall \ t > 0.
\]
Corollary

Assume $T = -\Delta$ and $\varphi_0 \in H^1(\mathbb{R}^3)$. Let $k \in \mathbb{N}$ and $A$ be a self-adjoint operator acting on $L^2(\mathbb{R}^3) \otimes s^k$. Assume that there exists $\theta \in [0, 1]$ such that $S_k^{-\theta/2} A S_k^{-\theta/2}$ can be extended to a bounded operator on $L^2(\mathbb{R}^3) \otimes s^k$ with operator norm $\| S_k^{-\theta/2} A S_k^{-\theta/2} \|$. 
Corollary

Assume \( T = -\Delta \) and \( \varphi_0 \in H^1(\mathbb{R}^3) \). Let \( k \in \mathbb{N} \) and \( A \) be a self-adjoint operator acting on \( L^2(\mathbb{R}^3) \otimes s^k \). Assume that there exists \( \theta \in [0, 1] \) such that \( S_k^{-\theta/2} \mathcal{A} S_k^{-\theta/2} \) can be extended to a bounded operator on \( L^2(\mathbb{R}^3) \otimes s^k \) with operator norm \( \| S_k^{-\theta/2} \mathcal{A} S_k^{-\theta/2} \| \).

(i) If \( \theta < 1 \), there exist \( C_k, D_k > 0 \) independent of \( N, t \) such that for any \( N \in \mathbb{N} \) with \( N \geq k \) and any \( t > 0 \) we have

\[
\left| \langle \Psi_{N,t}, \mathcal{A} \otimes I_{N-k} \Psi_{N,t} \rangle - \langle \varphi_{t}^{\otimes k}, \mathcal{A} \varphi_{t}^{\otimes k} \rangle \right| \leq C_k e^{D_k t} \frac{1}{N \min \left( \frac{1}{2}, 1-\theta \right)} \| S_k^{-\theta/2} \mathcal{A} S_k^{-\theta/2} \|. 
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(i) If $\theta < 1$, there exist $C_k, D_k > 0$ independent of $N, t$ such that for any $N \in \mathbb{N}$ with $N \geq k$ and any $t > 0$ we have

$$\left| \langle \Psi_{N,t}, A \otimes I_{N-k} \Psi_{N,t} \rangle - \langle \varphi_t \otimes^k, A \varphi_t \otimes^k \rangle \right| \leq \frac{C_k e^{D_k t}}{N \min(\frac{1}{2}, 1-\theta)} \|S_k^{-\theta/2} A S_k^{-\theta/2}\|.$$

(ii) If $\theta = 1$, then

$$\lim_{N \to \infty} \left| \langle \Psi_{N,t}, A \otimes I_{N-k} \Psi_{N,t} \rangle - \langle \varphi_t \otimes^k, A \varphi_t \otimes^k \rangle \right| = 0 \quad \forall t > 0.$$
Second main theorem
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Let for $r \in \mathbb{R}$

$$S_{k,r} := \sum_{i=1}^{k} (1 - \Delta x_i)^r.$$
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\]

**Theorem (Anapolitanos, Hott 2016)**

Let \( \Psi \in \mathcal{H}_N \cap H^s(\mathbb{R}^{3N}) \), \( \varphi \in H^s(\mathbb{R}^3) \) with \( \|\Psi\|_{L^2} = \|\varphi\|_{L^2} = 1 \) for some \( s > 0 \). Then we have for any \( \theta \in [0, 1) \) the estimate

\[
\text{Tr}\left| S_{k,\theta}^{1/2} (\gamma_N^{(k)} - P^{(k)}) S_{k,\theta}^{1/2} \right| \leq k C_{\Psi, \varphi, \theta, s} (a_N^{\min(1/2, 1-\theta)} + \|\gamma_N^{(k)} - P^{(k)}\|_{HS}^{1-\theta}),
\]

where \( C_{\Psi, \varphi, \theta, s} := 2 \max\{\|S_{1,s}^{1/2} \Psi\|_2 + \|S_{1,s}^{1/2} \varphi\|_2, (\|S_{1,s}^{1/2} \Psi\|_2 + \|S_{1,s}^{1/2} \varphi\|_2)^{2\theta}\} \).
Proof of the second Theorem

Abbreviate for all $r \in \mathbb{R}$

$$A_{k,r} := S_{k,r}^{\frac{1}{2}}(\gamma_{N}^{(k)} - P^{(k)}) S_{k,r}^{\frac{1}{2}}.$$

Proof is divided into three steps:
Proof of the second Theorem

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1. $\text{Tr}|A_{k,\theta s}| \leq 2\|A_{k,\theta s}\|_{HS} + \text{Tr}(A_{k,\theta s})$
Abbreviate for all $r \in \mathbb{R}$

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Proof is divided into three steps:

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3. $\text{Tr}(A_{k,\theta_s}) \leq k \max \left(\|S_{1,\frac{s}{2}} \Psi\|_2 + \|S_{1,\frac{s}{2}} \varphi\|_2, (\|S_{1,\frac{s}{2}} \Psi\|_2 + \|S_{1,\frac{s}{2}} \varphi\|_2)^{2\theta}\right) a_N^{\min(\frac{1}{2},1-\theta)}$
Step 1

\[ P_k \text{ is a rank-one projection in the } k\text{-particle space.} \]

Variational characterization of eigenvalues $A_k, \theta_s$ has at most one negative eigenvalue $\lambda_1$. Call the others ($\lambda_n$) $n \geq 2$ counting by multiplicity.

\[ \text{Tr} |A_k, \theta_s| = -2\lambda_1 + \sum_{n=1}^{\infty} \lambda_n \leq 2\|A_k, \theta_s\|_{HS} + \text{Tr} A_k, \theta_s. \]
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$P^{(k)}$ is a rank-one projection in the $k$-particle space.
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Step 1

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Step 2

Let \( L(x, y) \) be the integral kernel of \( \gamma(k) N - P(k) \) with \( x = (x_1, \ldots, x_k), y = (y_1, \ldots, y_k) \in \mathbb{R}^{3k} \).

\[ A_{k, \theta_\sigma} \] has an integral kernel given by

\[
\sum_{i=1}^{k} (1 - \Delta x_i) \theta_{\sigma} \leq \frac{1}{2} \left( \sum_{j=1}^{k} (1 - \Delta y_j) \theta_{\sigma} \right) L(x, y).
\]

Plancherel implies

\[
\| A_{k, \theta_\sigma} \|_{2, HS} = \int \left[ \sum_{i=1}^{k} (1 + |\xi_i|^2) \theta_{\sigma} \right] \left[ \sum_{j=1}^{k} (1 + |\eta_j|^2) \theta_{\sigma} \right] |\hat{L}(\xi, \eta)|^2 d\xi d\eta.
\]

Employ concavity of \( t \mapsto t^2 \) and H"older.
Step 2

$L(x, y)$ be the integral kernel of $\gamma_N^{(k)} - P^{(k)}$ with $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_k) \in \mathbb{R}^{3k}$. 
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$$\left[ \sum_{i=1}^{k} (1 - \Delta x_i)^{\theta s} \right]^{\frac{1}{2}} \left[ \sum_{j=1}^{k} (1 - \Delta y_j)^{\theta s} \right]^{\frac{1}{2}} L(x, y).$$
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Let \( L(x, y) \) be the integral kernel of \( \gamma_{N}^{(k)} - P^{(k)} \) with \( x = (x_1, \ldots, x_k) \), \( y = (y_1, \ldots, y_k) \in \mathbb{R}^{3k} \). \( \Rightarrow \) \( A_{k, \theta s} \) has an integral kernel given by

\[
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\]

Plancherel \( \Rightarrow \)

\[
\| A_{k, \theta s} \|_{HS}^2 = \int \left[ \sum_{i=1}^{k} (1 + |\xi_i|^2)^{\theta_s} \right] \left[ \sum_{j=1}^{k} (1 + |\eta_j|^2)^{\theta_s} \right] |\hat{L}(\xi, \eta)|^2 d\xi d\eta
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\|A_{k, \theta_s}\|_{HS}^2 = \int \left[ \sum_{i=1}^{k} (1 + |\xi_i|^2)^{\theta_s} \right] \left[ \sum_{j=1}^{k} (1 + |\eta_j|^2)^{\theta_s} \right] |\hat{L}(\xi, \eta)|^2 \, d\xi \, d\eta
\]

Employ concavity of \( t \mapsto t^\theta \) and Hölder.
Proof of (ii) in the first Theorem

Start again with

$$\text{Tr} \left| S_k^{\frac{1}{2}} (\gamma_{N,t}^{(k)} - P_t^{(k)}) S_k^{\frac{1}{2}} \right| \leq 2 \left\| S_k^{\frac{1}{2}} (\gamma_{N,t}^{(k)} - P_t^{(k)}) S_k^{\frac{1}{2}} \right\|_{HS} + \text{Tr} \left( S_k^{\frac{1}{2}} (\gamma_{N,t}^{(k)} - P_t^{(k)}) S_k^{\frac{1}{2}} \right).$$
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$$
\text{Tr}|S_k^{1/2} (\gamma_{N,t}^{(k)} - P_t^{(k)}) S_k^{1/2}| \leq 2 \|S_k^{1/2} (\gamma_{N,t}^{(k)} - P_t^{(k)}) S_k^{1/2}\|_{HS} + \text{Tr}(S_k^{1/2} (\gamma_{N,t}^{(k)} - P_t^{(k)}) S_k^{1/2}).
$$

Using $\frac{1}{N} \langle \Psi_{N,t}, H_N \Psi_{N,t} \rangle = \mathcal{E}(\varphi_0) = \mathcal{E}(\varphi_t)$,
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Using \( \frac{1}{N} \langle \Psi_{N,t}, H_N \Psi_{N,t} \rangle = \mathcal{E}(\varphi_0) = \mathcal{E}(\varphi_t) \), obtain

\[ \langle \Psi_{N,t}, -\Delta x_1 \Psi_{N,t} \rangle - \langle \varphi_t, -\Delta \varphi_t \rangle = -\frac{1}{2} \left( \langle \Psi_{N,t}, v_{12} \Psi_{N,t} \rangle - \langle \varphi_t \otimes^2, v_{12} \varphi_t \otimes^2 \rangle \right). \]
Proof of (ii) in the first Theorem

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$$\text{Tr}|S_k^{\frac{1}{2}} (\gamma_{N,t}^{(k)} - P_t^{(k)}) S_k^{\frac{1}{2}}| \leq 2\| S_k^{\frac{1}{2}} (\gamma_{N,t}^{(k)} - P_t^{(k)}) S_k^{\frac{1}{2}} \|_{HS} + \text{Tr}(S_k^{\frac{1}{2}} (\gamma_{N,t}^{(k)} - P_t^{(k)}) S_k^{\frac{1}{2}}).$$

Using $$\frac{1}{N} \langle \psi_{N,t}, H_N \psi_{N,t} \rangle = E(\varphi_0) = E(\varphi_t),$$ obtain

$$\langle \psi_{N,t}, -\Delta_{x_1} \psi_{N,t} \rangle - \langle \varphi_t, -\Delta \varphi_t \rangle = -\frac{1}{2} \left( \langle \psi_{N,t}, v_{12} \psi_{N,t} \rangle - \langle \varphi_t \otimes 2, v_{12} \varphi_t \otimes 2 \rangle \right)$$

$$\Rightarrow$$ Part (i) of the corollary applies!
For $\| \cdot \|_{HS} \to 0$, observe $\| S_k^{\frac{1}{2}} \gamma_{N,t}^{(k)} S_k^{\frac{1}{2}} \|_{HS} \leq \text{Tr}(S_k^{\frac{1}{2}} \gamma_{N,t}^{(k)} S_k^{\frac{1}{2}})$ and $\| S_k^{\frac{1}{2}} P_t^{(k)} S_k^{\frac{1}{2}} \|_{HS} = \text{Tr}(S_k^{\frac{1}{2}} P_t^{(k)} S_k^{\frac{1}{2}})$, i.e.,

$$\limsup_{N \to \infty} \| S_k^{\frac{1}{2}} \gamma_{N,t}^{(k)} S_k^{\frac{1}{2}} \|_{HS} \leq \| S_k^{\frac{1}{2}} P_t^{(k)} S_k^{\frac{1}{2}} \|_{HS}.$$
For \( \| \cdot \|_{HS} \to 0 \), observe \( \| S_k^{1/2} \gamma_{N,t}^{(k)} S_k^{1/2} \|_{HS} \leq \text{Tr}(S_k^{1/2} \gamma_{N,t}^{(k)} S_k^{1/2}) \) and 
\[
\| S_k^{1/2} P_t^{(k)} S_k^{1/2} \|_{HS} = \text{Tr}(S_k^{1/2} P_t^{(k)} S_k^{1/2}),
\]
i.e., 
\[
\limsup_{N \to \infty} \| S_k^{1/2} \gamma_{N,t}^{(k)} S_k^{1/2} \|_{HS} \leq \| S_k^{1/2} P_t^{(k)} S_k^{1/2} \|_{HS}.
\]

\( \rightarrow \) Argue via taking a weakly converging subsequence.
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- Natural assumption $\varphi_0 \in H^1$. [Lu12] requires $\varphi_0 \in H^3_A$ ($T = (i\nabla - A)^2$), [MS12] require $\varphi_0 \in H^2$ ($T = \sqrt{1-\Delta}$).
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- Interpolation result $\rightarrow$ better rates for smaller $\theta < 1$. First convergence result in the $(\theta = 1)$-case for $\varphi_0 \in H^1(\mathbb{R}^3)$!
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- Result can be extended to $T = (i \nabla - A)^2$ with $A \in L^2_{loc}(\mathbb{R}^3; \mathbb{R}^3)$ (\$\bowtie [Lu12]\$) and under some further assumptions also to $T = \sqrt{1 - \Delta} \to [AHH16]$, in progress.
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- Natural assumption $\varphi_0 \in H^1$. [Lu12] requires $\varphi_0 \in H^3_A$ ($T = (i \nabla - A)^2$), [MS12] require $\varphi_0 \in H^2$ ($T = \sqrt{1 - \Delta}$). $\|\varphi_t\|_{H^3_A}$ in the case $T = (i \nabla - A)^2$ resp. $\|\varphi_t\|_{H^2}$ in the case $T = \sqrt{1 - \Delta}$ both grow super-exponentially!

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- Result can be extended to $T = (i \nabla - A)^2$ with $A \in L^2_{loc}(\mathbb{R}^3; \mathbb{R}^3)$ (\cite{Lu12}) and under some further assumptions also to $T = \sqrt{1 - \Delta} \rightarrow [AHH16]$, in progress. However, the subcritical case $\lambda \leq -4\pi$ is not covered!
References

Thank you!