Want to study topology of smooth manifolds \( X \).

A powerful tool: introduce **Riemannian metric** \( g \) on \( X \).

Using \( g \), define **Laplacian operator** \( \Delta \) acting on forms:

\[
\Delta : \Omega^k(X) \to \Omega^{k-2}(X), \quad \Delta = \Delta_g = d^*d + dd^*.
\]

\( d^* \) is its adj. w.r.t. induced \( L^2 \)-P.

Consider the equation \( \Delta w = 0 \). This is a linear equation \( \Rightarrow \) space of sol.

\[
H^k_g = \ker(\Delta : \Omega^k(X) \to \Omega^k(X))
\]
is a vector space. \( \chi_{cpf} = \dim \ker(\Delta : \Omega^k(X) \to \Omega^k(X)) \) (ellipticity).

**Def.** \( b_k(X) = \dim \ker(\Delta : \Omega^k(X) \to \Omega^k(X)) \).

**Fact.** The integers \( b_k(X) \) (or \( b_k \)) depend on the choice of \( g \).

\( \Rightarrow \) they are invariants of the smooth manifolds \( X \).

**Fact.** There is a canonical iso. \( H^*_g(X) \cong H^*_g(X, \mathbb{R}) \) (singular cohomology)

\[
(\cong b_k(X) \text{ is inv. of the underlying topological mfd}).
\]

If \( X \) is oriented, a small refinement of \( b_{2n} \), \( H^*_g(X) \): Hodge \( \star \) operator: \( \star : \Omega^p \to \Omega^{n-p} \)

iere.

**Properties:**

1) \( \star \) is linear

\[
(\alpha \cdot \beta)_p = \alpha \cdot (\star \beta), \quad \alpha = \alpha^w + \alpha^c.
\]

2) \( [\star, \rho] = 0 \)

3) \( \star \) can decompose into \( \pm 1 \)-eigenspaces for \( \star : \mathbb{R}^{2n} = \mathbb{R}^{2n^+} \oplus \mathbb{R}^{2n^-} \)

4) \( H^n_g(X) = H^{n+}_c(X) \oplus H^c_n(X) \); \( b_{2n}(X) = b^+_c(X) + b^-_c(X) \).

**Ex:** What if \( \dim X = 4n + 2 \), \( a^2 = -1 \)?

Nonlinear eq.: \( \dim X = 4 \), Donaldson '80s.

Fix a Lie group \( G \), e.g. \( G = SU(2) \).
Fix \( G \)-bundle \( P \to X \), consider connections in \( P \).

**Locally**, a conn. in \( P \) is rep by a 1-form \( A \in \Omega^1(P, \mathfrak{g}) \) and has a curvature 2-form \( F = \omega^2(\mathfrak{g}) \).
Locally written: \( F = dA + A \wedge A \).

\( \neq 0 \) for \( SU(2) \) non-abelian.

Since \( F \) is a 2-form and \( \dim X = 4 \), can decompose \( F = \mathcal{F}^+ + \mathcal{F}^- \) (mod \( k \)).

Donaldson studies ASD YM equation

\[
\mathcal{F}^+ = 0 . \tag{11}
\]

(anti-selfdual)

Switch orient: ASD \( \leftrightarrow \) SD; in complex

adds don't want to swap orientations

\( \mathcal{F} \)

Claim: \( X \) oriented.

For \( \mathfrak{g} \) abelian, e.g. \( \mathfrak{g} = U(1) \), \( \mathcal{F} \) is linear.

For \( \mathfrak{g} \) non-abelian \( \mathfrak{g} \) non-abelian.

Consider instanton moduli space,

\[
M_p = \{ \text{connections } P \in \mathcal{A}(X) \} / \mathcal{G} .
\]

\( \mathcal{G} \) is a gauge group.

Example: If \( \mathfrak{g} = \mathfrak{su}(1) \) then \( \mathcal{M} = \mathbb{H}^2 / \mathbb{Z} \).

When \( \mathfrak{g} \) is non-abelian, \( \mathcal{M} \) is not a vector space. Still has some reasonable structure.

For \( \mathfrak{g} = \mathfrak{su}(2) \): \( P \) is classified by \( K = \int_X c_2(P) \in \mathbb{Z} \mod \mathcal{M} \).

Fact: If \( k > 0 \) and \( g \) is chosen generically, then \( \mathcal{M} \) is a finite-dim. mod.

\( \Gamma \) acts on \( M_k \).

\( \dim M_k = 8k - 2(1 - 2i/3)k^2 / k \).

\( X \) connected:

But \( \mathcal{M} \) un-connected (=) more to it than dim.

Donaldson introduced orientation on \( M_k \) (anomaly-free, \( F \in \mathfrak{su}(2) \mathcal{M}^+/k \)). \( \mathfrak{su}(2) \mathcal{M}^+/k \) labeled by classes \( \mathfrak{su}(2) \mathcal{M}^+/k \).

Donaldson invariants: schematically:

\[
\langle Q_k, \cdots, Q_k \rangle = \int_{M_k} Q_{k_1} \cdots Q_{k_k} \quad \mathcal{X} \neq \mathcal{Y} .
\]

Then: There are \( \# \mathcal{M}_k \), if \( g \), \( \int_X b_2^+(X) > 1 \).

Very powerful! But technically very hard; \( b_2^+(X) \) is hard, e.g. \( \mathcal{M}_k \) is not cpt.

(Powellson theory)
QFT and Donaldson invariants

1988: Witten found interpretation of \( \chi = \chi_1 \) in QFT, \( \chi \) of \( N = 2 \) supersym. YM theory with gauge group twisted \( \mathfrak{g} = SL(2, \mathbb{C}) \).

Imagined \( X \) to be "space-time" and computing experimental measurements ("correlation terms")
According to Witten, QFT, this means computing \( n \)-dim. integrals, of the form,

\[
\left< \mathcal{O}_N \right> = \int \mathcal{D} \phi \mathcal{O}_N \phi \quad , \quad \mathcal{S} : 2 \rightarrow \mathbb{R} \text{ action}, \quad \mathcal{O}_2 : 2 \rightarrow \mathbb{R} \text{ observable}
\]

\( \mathcal{L} \equiv \text{"space of fields" (like } x^{\alpha}(x), \phi^{\alpha}(x), \text{ form of } \mathbb{R} \times X) \).

In gen., these \( \left< \mathcal{O}_N \right> \) are hard. But, in SYM, Witten found localization:
reduces it to the \( S \) Donaldson considered!

Effective field theory: Seiberg-Witten 1994: answered a fund. Q about SYM:

\[ \text{How does the theory behave at low energies?} \]

To study pond:

1) "High energy": \( 10^{10} \) particles
2) "Low energy": NSE

What SW did:

1) "High energy": \( \mathcal{N} = 2 \) SYM, \( G = SU(2) \) (\( \text{and ASD YM eq.} \))
2) "Low energy": roughly \( \mathcal{N} = 2 \) SYM, \( \mathfrak{g} = U(1) \) coupled to matter-particles (\( \text{SW equations} \))

Effort

SW eq.:
- connection in \( U(1) \)-bundle \( E \) w/ line of a Spin\( ^c \)-structure.

\[
\nabla^{\psi} g(\psi, \nabla^{\psi}) \quad \text{and} \quad D = \text{Dirac op. } \quad g : S^{\nu} \otimes S^{\nu} \rightarrow \bigwedge^2 T^* X.
\]

One prelim. indication that SW eq. are easier than ASDYM:

\[
\mathcal{M} \ni \{ \text{pairs (}\text{comp. } \mathcal{M}, \text{ obey (\psi, \nu))} \} / \mathcal{G}
\]
is \( \mathcal{G} \) mild for generic \( g \).
Our goals: Starting w/ QFT, where dim \( k = 0 \), then dim \( k = 1 \) (2H)
\[ \text{dim } k = 2 \text{ very briefly} \]
\[ \text{dim } k = 4 \]

**Susy**

**Localization**

**Effective field theory (high energy vs low energy)**

Ex.: \( M = (A, D \text{ connections}) / (\text{gauge group}) \)

If \( G = U(1) \), show \( M \cong \mathbb{R} / (k) \) (not exactly).

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Sept. 5 2017

Today (ThD): Passage from fundamental to ref. in 0-dim. QFT.

Recall: QFT involves integrals over a space \( \mathbb{R}^D \), some kind of free space.

Let's take \( k = 0 \), \( \mathbb{R}^D \subseteq \mathbb{R} \). Define action \( S : \mathbb{R}^D \to \mathbb{R} \)

\[
S(x) := \frac{m}{2} x^2 + \frac{\lambda}{4!} x^4, \quad m > 0, \lambda > 0.
\]

Now we can define partition func:

\[
\mathcal{Z} := \int dx e^{-S(x)}
\]

**Observables:** Fok. func: \( f : \mathbb{C}^D \to \mathbb{R} \).

(Unnormalized expectation value)

\[
\langle f \rangle := \int dx f(x) e^{-S(x)}
\]

Both func at \((\lambda, m)\). How to compute?

Start w/ \( \mathcal{Z} \):

If \( x = 0 \), then

\[
\mathcal{Z}_0 := \mathcal{Z}(m, \lambda = 0) = \sqrt{\frac{\mathcal{Z}}{m}}.
\]

\[
\mathcal{Z}(m, \lambda) = \int dx e^{-\frac{m}{2} x^2 - \frac{\lambda}{4!} x^4} \text{ try to expand around } \lambda = 0 \text{. Expand powers:}
\]

\[
\mathcal{Z} = \int dx \sum_{n=0}^{\infty} \left( \frac{\lambda}{4!} \right)^n e^{-\frac{m}{2} x^2} = \sum_{n=0}^{\infty} \left( \frac{\lambda}{4!} \right)^n \int dx \frac{\alpha^n}{n!} e^{-\frac{m}{2} x^2}.
\]

Now use:

\[
\int dx x^2 e^{-\frac{m}{2} x^2} = \sqrt{\frac{2\pi}{m}} \frac{1}{n!} \left( \frac{\pi \lambda}{2} \right)^{\frac{n}{2}} (4^n/n!(2\pi)^{\frac{n}{2}}) (E_x)
\]

So,

\[
\mathcal{Z}(m, \lambda) = \sqrt{\frac{2\pi}{m}} \sum_{n=0}^{\infty} \left( \frac{\lambda}{4!} \right)^n \frac{(4^n n!)^{\frac{n}{2}}}{n! (2\pi)^{\frac{n}{2}}} \.
\]

\[
\mathcal{Z} = \frac{\sqrt{\pi}}{m} \left( 1 - \frac{\lambda}{8} \lambda + \frac{3\lambda^2}{24} \lambda^2 + \ldots \right).
\]
Given \( f : \mathbb{R}^+ \to \mathbb{C} \) the formal series \( \sum_{n=0}^{\infty} c_n t^n \) is an asymptotic series for \( f \) as \( t \to 0^+ \) if:

\[
\lim_{t \to 0^+} t^N \left| f(t) - \left( \sum_{n=0}^{N} c_n t^n \right) \right| = 0.
\]

Then we write \( f(t) \sim N \sum_{n=0}^{\infty} c_n t^n \).

This means:

\[
\lim_{t \to 0^+} \frac{f(t) - c_0}{t^N} = 0,
\]

\[
\lim_{t \to 0^+} t^{N-1} \left( f(t) - c_0 - t c_1 \right) = 0.
\]

**Prop.** The series \( (x) \) is an asymptotic series for \( Z(m, \lambda) \) as \( \lambda \to 0^+ \).

**Prop.** If \( f \) has a conv. Taylor series, then it's an asymptotic series.

**Prop.** Any \( f(t) \) can have at most one asymptotic series as \( t \to 0^+ \).

**Feynman Diagram:** Let's rewrite \( Z(m, \lambda) \) one more time:

\[
Z(m, \lambda) \sim \sqrt{\frac{2\pi m}{\lambda}} \sum_{n=0}^{\infty} \frac{\epsilon_{m,n}!}{(2\pi)^{2n}} \frac{\lambda^n}{n!^m}.
\]

**Basic Object:**

\[
\chi \quad \text{vertex w/ 4 1-edges attached}
\]

**Feynman diagrams for \( S(\lambda) \):** Place some # of vertices, connect up the \( \frac{1}{2} \) edges.

\[
S(\lambda) = \frac{\sqrt{\pi} \lambda^{2m}}{2^{m+1} m!}
\]

Let \( D_{2n} = \{ \text{diagrams w/ } n \text{ vertices} \} \)

**Prop.** The # of ways to pair up \( 2k \) objects is \( \frac{(2k)!}{k! \cdot 2^k} \).

\[
S_n = D_{2n} \cdot h_n, \quad h_n = (S_0)^n \times S_n
\]

**Prop.** The # of ways to pair up \( 2k \) objects is \( \frac{(2k)!}{k! \cdot 2^k} \).
$$Z_{(n)} = \sqrt{\frac{2\pi}{\kappa}} \sum_{k=0}^{\infty} \left(\frac{\lambda}{8}\right)^k \frac{\Gamma(k+1)}{k!} \cdot \left(\frac{1}{4\mu_1} \cdot \sum_{1D=4} \frac{1}{6^{22}} \cdot \frac{1}{141} = 2 \cdot \frac{1}{141} = 2\right)$$

Orbit-stab. then \( \frac{Z_{(n)}}{Z_0} \) \( \sum (-\lambda)^k \sum_{\pi \in \text{Aut } G} \frac{1}{|\pi|} \in \mathbb{C} / \mathbb{L} \)

\[ \frac{Z_{(n)}}{Z_0} = \sum_{\pi \in \text{Aut } G} \left(\frac{-\lambda}{8}\right)^{\text{vertices}(\pi)} \frac{1}{|\text{Aut } G|} \]

\( \uparrow \) Feynman rules:

- Draw one rep. \( \Gamma \) in each equiv. class. Define weight \( w_\pi \) as \( \prod \text{ factors} \)

\[ \text{divided by } |\text{Aut } G|, \]

\[ \frac{Z}{Z_0} = 1 + \frac{(-\lambda)}{8m^2} + \frac{\lambda^2}{8^2 \text{m}^4} + \frac{\lambda^4}{8^3 \text{m}^6} + \frac{\lambda^5}{8^4 \text{m}^8} + O(\lambda^9) \]

\[ \frac{Z}{Z_0} \approx 1 + \frac{-\lambda}{8m^2} + \frac{5\lambda}{384 \text{m}^4} + \frac{\lambda^2}{8^2 \text{m}^6} + \]

\[ \sum_{\pi \in \text{Aut } G} w_\pi = \exp \left( \sum_{\pi \in \text{Aut } G} w_\pi \right), \]

i.e.,

\[ \log \left( \frac{Z_{(n)}}{Z_0} \right) = \sum_{\pi \in \text{Aut } G} w_\pi. \]

**Prop:** \( \sum \) over connected diagrams is related to \( \sum \) over all diagrams by

\[ \sum_{\pi \in \text{Aut } G} w_\pi = \exp \left( \sum_{\pi \in \text{Aut } G} w_\pi \right), \]

i.e.,

\[ \log \left( \frac{Z_{(n)}}{Z_0} \right) = \sum_{\pi \in \text{Aut } G} w_\pi. \]

\[ \text{for energy} \]

\[ \sum_{\pi \in \text{Aut } G} w_\pi \]

**To compute** \( \langle x^k \rangle = \int x^k e^{-S} dx \) **are similar** \( \sum \) over Feynman diagrams w/ a new 1-valent vertex. Requiring diagram has exactly \( k \) of these (fixed by autom. of \( \Gamma \))

To compute normalized e.v.:

\[ \langle x^k \rangle = \sum \text{diag. w/ add. rule; every conn. component of } \Gamma \text{ must have at least } 1 \text{ 1-valent vertex.} \]
\[ \frac{\langle x^2 \rangle}{\langle x \rangle} = \frac{1}{m} - \frac{\lambda}{2m^2} + \ldots \]

3. More gen., \( S = \frac{m}{2} x^2 + \sum_{k=3}^{\infty} \frac{\lambda_k}{k!} x^k \).

\[ \begin{array}{cccc}
1 & \Lambda & x & \ldots \\
m & -\lambda_3 & -\lambda_4 \\
\end{array} \]

\[ \frac{Z}{Z_0} = 1 - \frac{\lambda_4}{8m^2} + \frac{\lambda_3^2}{12m^2} + \ldots \]

4. Take \( Z = \frac{1}{N!} \sum_{\text{diagram}} \frac{1}{4!} C_{ijk} x^i x^j x^k x^l \).

Then \( Z_0 = \frac{(2\pi)^{N/2}}{\sqrt{\det H}} \int_{\mathbb{R}^N} e^{-\frac{1}{2} x^T H^{-1} x} \).

\[ \frac{Z}{Z_0} \text{ computed by Feynman rules using edges labeled by } i = 1, \ldots, N \]

\[ \begin{array}{c}
\frac{1}{(H^{-1})^i} \quad \Lambda \quad x^l \\
\end{array} \]

\[ -C_{ijk} \]

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\[ \text{Sept. 7, 2017} \]

Last time: field space \( C = \mathbb{R}^N \), \( S = \frac{1}{2} x^T H x + \frac{1}{4!} C_{ijk} x^i x^j x^k x^l \)

To compute asympt. series as \( C \to 0 \)

\[ \sum \text{ over Feynman diagrams} \]

To determine weights sum over labels of \( \frac{1}{2} \)-edges by \text{inf}.
Or:
\[ z = v, \quad M \in \text{Sym} 2 V \wedge 1, \quad C \in \text{Sym} 4 V \wedge 2 \]

Solution:
\[ S(x) = \frac{1}{2} M(x, x) + \frac{1}{4} C(x, x, x, x), \quad S: C \rightarrow \mathbb{R} \]

A coupled system: Now say \( z = \mathbb{R}^2 \), coords \( x, y \),
\[ S(x, y) = \frac{\mu}{2} x^2 + \frac{\mu}{2} y^2 + \frac{\mu^2}{4} x y \]

Say we're "really interested" in \( x \). We want to compute \( \langle x^5 \rangle \), not \( y \) or \( xy \).

Feynman rules,
\[
\begin{align*}
\log \left( \frac{2}{2} \right) & = \infty + \ldots + \infty \quad \ldots + \quad \infty \\
\log \left( \frac{3}{2} \right) & = -\frac{\mu}{4mM} + \frac{\mu^2}{16m^2M^2} + \frac{\mu^2}{8m^2M^2} + \ldots \quad \ldots \quad \ldots \\
\langle x^5 \rangle & \approx \frac{1}{5} + \ldots + \frac{1}{3} + \ldots + \ldots + \ldots \quad \ldots \\
\langle x^5 \rangle & \approx \frac{3}{5} - \frac{3\mu}{2m^3M} + \frac{3\mu^2}{4m^4M^2} + \ldots \quad \ldots \quad \ldots \\
\end{align*}
\]

If we only want \( \langle x^5 \rangle = \int dx \ dy \ x^5 e^{-S(x,y)} \).

By Fubini, we can define \( S_{\text{eff}}(x) = \int dy \ e^{-S(x,y)} = e^{-S_{\text{eff}}(x)} \).

In this theory, we can compute \( S_{\text{eff}}(x) \) (or at least its asymptotic series as \( \mu \rightarrow 0 \))
\[
S_{\text{eff}}(x) \approx \frac{\mu}{2} x^2 + \frac{\lambda_k}{k+3} x^k \quad k \geq 3
\]

With \( \mu = m + \frac{\mu}{2M} \) \( m \rightarrow \infty \) \( \mu \rightarrow 0 \):
\[
\lambda_k = \begin{cases} 0 & k \text{ odd} \\ (-\frac{M}{\mu})^{k/2} \frac{1}{k+2} & k \text{ even} \end{cases}
\]
Symmetries

Let's get back to $\tau = \tau_1 \leq \frac{\tau_4}{2} < x^4$. We have

$$\langle x^n \rangle = 0 \text{ for } n \text{ odd.}$$

One way to see it:

$$\langle x^n \rangle = \int dx \ x^n e^{-x} = \int dx \ x^n e^{-x} x^{-1} e^x = \int dx \ (x^{-1} e^{-x})' \ e^{-x} = \int dx \ e^{x} = \langle e^{x} \rangle$$

In particular: No Feynman diagrams for odd $n$ and $F$-vertex vertices.

Prop. If $\tau < \tau$ and the measure $\tau$ is invariant under a group $G$, then

$$\langle \theta \rangle = \langle \theta \theta \rangle,$$

where $\theta \in \mathbb{R}$ any observable, $\theta \theta = g \theta$.

If $G$ is Lie group, diff. $\theta \to g \exp(tX), X \in G$ take $\frac{d}{dt} |_{t=0}$, get $\langle \theta \theta \rangle = 0$.

So far: output of our computations are tens of pages, e.g. (3.12). Not deformation invariant.

To get things that are deformed, need $\tau$ more ingredient: fermions.

Replace $\tau$, which was a $\mathbb{Z}_2$-graded $u$-space, by a supermanifold (super vector space).

Def. A super vector space is a $\mathbb{Z} / 2$-graded $u$-space $V = V^0 \oplus V^1$.

Ex.: $V^0 = \mathbb{R}^0, V^1 = \mathbb{R}^1$, then $V$ is called $\mathbb{R}^{0|1}$.

Def.: The symmetric monoidal category of super vector spaces is the same as for ordinary $\mathbb{Z}/2$-graded $u$-spaces except $S: V \otimes W \to W \otimes V$ usually $v \otimes w \mapsto w \otimes v$.

We take $V \otimes W \to (-1)^{\dim(W)\dim(W)} w \otimes v$, $v \in V^1, w \in W^1$, i.e., we have extra - if $v \otimes w 
\in V^1 \otimes W^1$.

Symmetric algebra: $\text{Sym}^* V := T^* V / \langle v \otimes w - w \otimes v \rangle$.

If $V = V^0$, then $\text{Sym}^* (V)$ is the usual $\text{Sym}^* (V^0)$.

If $V = V^1$, then $\text{Sym}^* (V) = \Lambda^*(V^1)$. 
Def.: Given a super vector space $V$, the algebra of polynomial forms on $V$ is denoted $O(V) = \Sigma_{n \geq 0} (V^n)$. $O(V)$ itself is a super vector space, even a (super) commutative algebra.

In QFT, we want a space $\mathcal{C}$ and a "function on $\mathcal{C}$". If $\mathcal{C}$ is a supermanifold, then $\mathcal{O}(\mathcal{C})$

A fermionic theory

Take $\mathcal{C} = \mathbb{R}^{0|2}$. $\mathbb{C}$ has "coord. basis" $\{\psi^1, \psi^2\}$. $\mathcal{O}(\mathbb{C})$ has basis $\{1, \psi^1, \psi^2\}$, $\Lambda^0(\mathbb{C})$ has basis $\{\psi^1, \psi^2\}$.

Let's take the action functional $S = \frac{1}{2} \int \psi^1 \psi^2$. (Rem.: $S^2 \geq 0$)

We'd like to make sense of $Z = \int d\psi e^{-S}$.

Rules for integrating over odd variables:

- over $\mathbb{R}^n$, $\int d\psi (a\psi + b) = a$.
- $\int d\psi \psi^1 \psi^2 \ldots d\psi^k \psi^l = \int d\psi \int d\psi^1 \ldots \int d\psi^k \psi^l \ldots$.
- So, $Z = \int d\psi^1 d\psi^2 e^{-S} = \int d\psi^1 d\psi^2 (1 - \frac{1}{2} \int d\psi^1 \psi^2) = \frac{1}{2} \int d\psi^1 (d\psi^1)^2 \psi^2 = \frac{1}{2} Z$.

NB: Compare to what we get for even fields:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{\sqrt{1!}}$$

for fermions, $\int_{-\infty}^{\infty} e^{-\frac{1}{2} \int d\psi d\psi} = \frac{1}{\sqrt{2\pi}}$.

Last time: Odd QFT where $\mathcal{C}$ is an odd super.

Ex.: $\mathcal{C} = \mathbb{R}^{0|2}$ two coordinate fields $\psi^1, \psi^2 \in \mathcal{O}(\mathcal{C})$.

Even fields $1, \psi^1, \psi^2 \in \mathcal{O}(\mathcal{C})$.

Even fields:

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2} \int d\psi d\psi} = \frac{1}{\sqrt{2\pi}}$$

$S = \int_{-\infty}^{\infty} \frac{1}{2} (\psi^1 \psi^2 - \bar{x} \bar{y})^2$

$S + \frac{1}{2} \Sigma_{n \geq 0} (V^n)$
\[ z = \int \sigma \, d\nu - s = \int \sigma \, d\nu \left(1 - \frac{1}{2} H\psi \cdot \psi\right) \]

**Def.:** A (translation-invariant) measure on an odd super-vsp. \( V = V' \) is \( \psi^* \in \Lambda^{top}(\Pi V) \)

\[ \text{parity change: } \bar{\Pi}(V \otimes V') = V' \otimes V \]

**Def. (integral over odd vsp.)**

For \( f \in \Theta(V) = \Lambda^{top}(\Pi V)^* \) let \( \phi \in \Lambda^{top}(\Pi V')^* \) be top comp., then

\[ \int \sigma \, d\mu = \langle \sigma, \phi \rangle \quad (\sigma \in V) \]

**Ex.:** say \( V = \mathbb{R}^\otimes \) odd and \( p \). Show \( \exists \) measure \( d\nu \) on \( V \) s.t.

**Var. 2:** \( \int d\nu \left( a \psi + b \right) = a \)

We call this measure \( d\nu \).

\[ \int d\nu = 1, \quad \int d\nu \psi = 0. \]

**Ex.:** say \( V = \mathbb{R}^\otimes \) c.e.R. Show \( \exists \) measure \( c \, d\nu \) and \( d(\psi \nu) \) on \( V \) s.t.

\[ \int \psi \, d(\psi \nu) = \psi \int d\nu \psi \]

The prob.: \( d(\psi \nu) = \frac{1}{2} d\nu \)

\[ (\int d(\psi \nu) \psi \nu = 1) \]

Similarly on \( R^{\otimes n} \) define \( d\nu = d\nu \psi \cdot d\nu \cdot d\nu \cdot \ldots \cdot d\nu \) by

\[ \int d(\nu \otimes \ldots \otimes \nu) = 
\int d(\nu \otimes \ldots \otimes \nu) = \frac{1}{n!} \]

Now,

\[ Z = \int d(\nu \otimes \ldots \otimes \nu) \left( 1 - \frac{1}{2} H\psi \cdot \psi \right) = \frac{1}{2} M. \]
More generally, let $\bar{\psi}$ be any odd superfield with
\[
M \in \text{Sym}^2 (V^*) = \Lambda^2 ((\bar{\psi} \psi)^*)
\]
\[
C \in \text{Sym}^4 (V^*) = \Lambda^4 ((\bar{\psi} \psi)^*)
\]
Then one can take
\[
S = \frac{1}{2} M + \frac{1}{4!} C \in C \mathcal{O}(c).
\]
Or after choosing a basis for $\Pi V^4$, i.e., $V \cong \mathbb{R}^{1,4}$
\[
S = \frac{1}{2} M_{IJ} \psi^I \psi^J + \frac{1}{4!} C_{IJKL} \psi^I \psi^J \psi^K \psi^L.
\]
$M_{IJ}$ is antisymmetric. The matrix $C_{IJKL}$ is antisymmetric, tensor.
$Z = \sum \mu e^{-S}$ can be evaluated algebraically.
\[
\text{Ex. : Say } Z = R^{014} \text{ and } S = m \psi \psi + m \psi \Psi^I \gamma^4 \lambda \bar{\psi} \bar{\psi} \gamma^I \tau \phi \phi.
\]
\[
\Rightarrow Z = m^2 - \lambda.
\]

Perturbation theory: Feynman diagram expansion w/ fermions; just like the rules w/ bosons except for some extra signs.

In this theory (e), get
\[
Z_0 = m^2 \quad \text{in general } \mathcal{P}(H)
\]
\[
\frac{Z}{Z_0} = (\text{empty}) + \mathcal{A} + \mathcal{B} + \mathcal{C} + \ldots
\]
For weight of each diagram, $\sum$ over labels $I = 1, \ldots, 4$ of linear
\[

\begin{array}{c}
\begin{array}{c}
\mathcal{F}
\end{array}
\end{array}
\]

($\mathcal{M}^*)$ $I \bar{\psi}
\]

$C_{IJKL}$
Say \( T = \nu = \nu^\partial \nu^! \). Need an \( S \)-theory for such \( \nu \).

Define
\[
\nu^{\partial \partial} (\nu) := T \vee \nu^\partial \nu^! \otimes \nu^0.
\]

**Def.** The 
**Berezinian line of a super** \( \nu \times \nu \), \( \nu \) is
\[
\text{Ber}(\nu) := \Lambda_{\text{top}}^\nu \nu^\partial \big( \Lambda_{\text{top}}^\nu (\nu^\partial) \big)^*.
\]
An element \( d\mu \in \text{Ber}(\nu) \) is an integration measure.

**Def.** If \( \nu \) is an oriented super \( \nu \times \nu \), \( \nu = \wedge \nu^0 \wedge \nu^0 \), \( \nu = \nu^0 \), then
\[
\int \nu \wedge \nu^0 = \int \nu \wedge \nu^0 \big( \nu^0 \nu^0 \big)^*.
\]

On \( \mathbb{R}^n \) have canonical
\[
d\nu = d\nu^\partial d\nu^0 = (dx^1 \ldots dx^n) \otimes (dp^1 \ldots dp^n).
\]

Take \( \nu = \nu^{1\!\!2} \)
\[
S(\nu) = S_1(\nu) + S_2(\nu)\nu^0 \nu^0.
\]
\[
Z = \int \nu \wedge \nu^0 \big( \nu^0 \nu^0 \big)^* = \int \nu \wedge \nu^0 \nu^0 = \nu^0 \nu^0 = \langle \nu \nu^0 \rangle \text{ in the even theory } \nu = \nu^0.
\]

For general \( S_1, S_2 \) nothing interesting about this. One special case is much better:

Fix some \( \ell : \mathbb{R} \to \mathbb{R}^1 \), \( \ell(x) = \hat{x} \) as \( x \to \infty \) and set \( S_1(\nu) = \frac{1}{2} \nu \nu^0 \), \( S_2(\nu) = \nu^0 \nu^0 \),
\[
\text{so } S = \nu \nu^0 + \nu^0 \nu^0.\nu^0.\nu^0. This \ S \text{ is invariant under a certain odd vector field on } \nu.
\]

**Def.** If \( \mathfrak{a} \) is a comm. superalgebra (e.g. \( \mathfrak{g} = \mathfrak{g} (1) \)), then \( \mathcal{D} : \mathfrak{a} \to \mathfrak{a} \) (even or odd) is a

**derivation** on \( \mathfrak{a} \) if
\[
\mathcal{D} (a b) = (\mathcal{D} a) b + (-1)^{||a||} a \mathcal{D} (b)
\]
Def.: If $V$ is a super v.s., let $\text{Vec}(V)$ be the space of all derivations of $\text{O}(V)$. $\text{Vec}(V)$ is a super v.s.

Ex.: $\text{Vec}(V)$ is a super Lie algebra.

On $\mathbb{R}^{m|m}$ have usual $\partial_x \in \text{Vec}^0(V)$ now also $\partial_y \in \text{Vec}^1(V)$ defined by $\partial_y (x^i y^j) = 0$, $\partial_y (y^k) = y^k$, $\partial_y (y^k y^l) = -y^l$.

$S = \frac{1}{3} x^i x^j x^k$ is invariant under 2nd v. fields $\partial_1, \partial_2$:

$$\partial_1 : x^i \partial_x + \frac{d}{d\xi} \partial_y$$
$$\partial_2 : y^k \partial_x - \frac{d}{d\xi} \partial_y$$

Ex.: $[\partial_1, \partial_2]$ is an even v. field $X$ with $X^2 = 0$. What's $X^2$?

$\partial_1, \partial_2$ are also divergence-free:

Def.: Lie derivative of a section of $\text{Vec}(V)$ along a v. field

$$\mathcal{L}_X : \frac{x^i \partial_x + \frac{d}{d\xi} \partial_y}{dx^i \partial_y}$$

$\text{Deligne - Huybrechts - Lehn}$]

Lemma: If $Q$ is a divergence-free $(\int_{\mathcal{Q}, \delta \mu} = 0)$ v. field (even or odd) on a super v.s. V w/ no ghost $\delta \mu$ and $f \in \mathcal{C}^\infty(V)$, then

$$\int_{\mathcal{V}, \delta \mu} (Q \ast f) = 0.$$

Proof:

$$Q = x^i \partial_x + \frac{d}{d\xi} \partial_y$$

$$\int_{\mathcal{V}, \delta \mu} Q \ast f = \int_{\mathcal{V}, \delta \mu} (Q \ast f) \text{top} = \int_{\mathcal{V}, \delta \mu} x^i \partial_x (Q \ast f) \text{top}$$

$$= \int_{\mathcal{V}, \delta \mu} (x^i \partial_x q + \frac{d}{d\xi} \partial_y q \text{top})$$

= 0 b/c $Q$ div-free
Prop.: say \( V \) super v.s. \( u \) \( \frac{dv}{dt} \) div. free odd v-field on \( V \) \( \partial [0, 0] = 0, i \partial S = 0. \)

\( \int \phi_t \) smooth family of odd \( \phi_t \in \mathcal{E}_c^p(V), \phi_0 = 0. \)

Let \( S_t = S + \partial \phi_t \). Then \( \tau_t^t \) is \( t \)-indep. \( (\tau_t^t = \tau_{t0}). \)

\[ \begin{align*}
\partial_t \tau_t^t &= -\int \phi_t (Q \phi_t) e^{-(S + \partial \phi_t)} e - \int d\mu (Q \phi_t) e^{-S - \partial \phi_t} e = -\int d\mu (Q \phi_t) e^{-S - \partial \phi_t} e - \int d\mu (Q \phi_t) e^{-S - \partial \phi_t} e = 0.
\end{align*} \]

\( \therefore \)

\[ \begin{align*}
\tau_t^t &= \langle Q \phi_t \rangle, \langle X_0 \rangle = 0
\end{align*} \]

\( Z = \mathbb{R}^1 \quad h : \mathbb{R} \to \mathbb{R}, S = \int_{\tau} \tau_t^t + \frac{\partial \phi_t}{\partial \phi_t} \phi_t \phi_t \\
\tau_t^t &= \int e^{-S} e = Q_1 = \psi_1 \phi_t + \frac{\partial \phi_t}{\partial \phi_t} \phi_t \phi_t, \\
Q_1 &= \psi_1 + \frac{\partial \phi_t}{\partial \phi_t} \phi_t \phi_t,
\end{align*} \]

\[ \begin{align*}
\psi_t = \partial_t \psi_t = \partial_t \phi_t = \partial_t (\psi_1 + \psi_2 \phi_t + \frac{\partial \phi_t}{\partial \phi_t} (\partial_t \phi_t - \partial_{\phi_t} \phi_t))
\end{align*} \]

\[ \begin{align*}
\{\psi_1, \phi_t\} &= 0 \quad \frac{\partial \phi_t}{\partial \phi_t} \phi_t \phi_t
\end{align*} \]

Last time. \( \Rightarrow \) \( \tau_t^t \) is inv. under \( C_c \) of deformations of \( S_t \), i.e., if \( \tau_t^t = \tau_{t0} \), \( \tau_{t0} = f = \phi_t \phi_t \), then \( \partial_t \tau_t^t = 0, \)

\[ \begin{align*}
\phi_t^0 \phi_t \phi_t (c)
\end{align*} \]

Consider deforming \( \theta(x) \) to a family \( \tilde{\theta}(x) \) and \( S(\theta) \) deforms to \( S(\tilde{\theta}) \).

\[ \begin{align*}
\tilde{s}_t^t (x) = s_t^t e^{\tilde{s}_t^t(x)} \psi_1 \phi_t \phi_t
\end{align*} \]

But \( s_t^t (x) = Q_1 \phi_t \phi_t = -\frac{\partial \phi_t}{\partial \phi_t} \).

So Prop. \( \Rightarrow \) \( \tau_t^t \) dest. does not depend on \( \tilde{\theta}(x) \).

\( \tau_t^t \) is inv. under cyclic supported variations of \( \phi(x) \)

Bootstrap this to \( \tau_t^t \) only depends on \( \phi \) at the limit \( \phi(x) = \phi(x) \).

\[ \begin{align*}
\begin{aligned}
\text{Prop.} &\Rightarrow \tau_t^t \text{ dest. does not depend on } \phi(x).
\end{aligned}
\end{align*} \]
Cl. Donaldson theory: $2$ inv. indep. $g$ if $s_2^+(x) > 1$ if $s_2^+(x) = 1$ "wold ring".

Localization

Let's take $F(x) \to x \cdot (C(x))$, $\lambda \in \mathbb{R}^+$. $Z$ is indep. of $\lambda$. Compute in limit $\lambda \to \infty$

Recall how to study ordinary integrals like $\int_{-\infty}^{\infty} e^{-\lambda x^2} \, dx$ as $\lambda \to \infty$.

$\lambda$-asymptotics of $F(x)$ controlled by steepest descent method.

Prop.: As $\lambda \to \infty$, $\int F \lambda$ has a global minimum at $x_c$

$$\int_{-\infty}^{\infty} e^{-\lambda F(x)} \, dx \approx e^{-\lambda F(x_c)} \frac{\sqrt{\pi}}{N^{1/2}(x_c)} e^{-\lambda F(x_c)}$$

[Bender - Orszag]

\[ 1 \quad \text{as} \quad x \to \infty, \quad \lim_{x \to \infty} \frac{F(x)}{x} = 1 \]

:: i.e., it's ok to truncate $F(x)$ to quadratic order around $x_c$.

Let's make the same kind of quadratic approx. around each critical point $x_c$, $F(x_c) = 0$, for our action $S$

$$2 \lambda \mathcal{S}(\lambda) \approx \sum_{x_c} \int dx \, d\vec{p} \, e^{-\frac{1}{2} \lambda \mathcal{G}'(x_c)^2 (x-x_c)^2 - \lambda \mathcal{G}'(x_c) \vec{p} \cdot \vec{p}}$$

Inv. for $\lambda$

$$\lambda \cdot x_c - \lambda \cdot x_c^2$$

get equality

$$= \sum_{x_c} \frac{2 \pi}{\sqrt{\lambda \cdot x_c^2}} \mathcal{G}'(x_c) = \sqrt{2 \pi} \sum_{x_c} \frac{e^{1/2}}{\lambda \cdot x_c} = \sqrt{2 \pi} \sum_{x_c} f_n \left( \mathcal{G}'(x_c) \right)$$

Eqn.

$$\frac{d^2}{dx^2} = -1 \quad \text{with} \quad \frac{2}{\sqrt{2 \pi}} = 0$$

Get

$$\frac{2}{\sqrt{2 \pi}} = \frac{1}{2} (3 + \sqrt{3 - 3 \sqrt{2}})$$
Localization in a $O(d,\mathbb{R})$ model

Say $(M,\omega)$ be cpt sympl. md $d\omega=0,\omega^d=0$, dim $M=2n$.

generated by v. field $\psi \in \Omega^1 M$, $Y=\omega^{-1}(dH)$, $H: M \to \mathbb{R}$.

Say all fixed pts of $Y$ are isolated. Fix $x \in \mathbb{R}$. We study

$$\int_{M} \frac{\omega^n}{m^n} e^{i\chi H}$$

Ex. $M=S^1$, $\omega=\sin \vartheta d\vartheta d\varphi$, $Y=d\varphi$.

$$H=2=\cos \vartheta$$

$$\int_{M} \frac{\omega^n}{m^n} e^{i\chi H} = \int_0^1 \sin \vartheta d\vartheta d\varphi e^{i2\cos \vartheta} = 2\pi \int_0^{\pi} e^{i2\cos \vartheta} \sin \vartheta d\vartheta = -2\pi \int e^{i2\cos \vartheta} d\vartheta = -4\pi \sin \vartheta \int e^{i2\cos \vartheta} d\vartheta$$

This answer exhibits localization near $x$ of contributions

$$\ell = \frac{\pi}{\sqrt{N}} \cdot e^{iNkH(x)}$$

$x_c$ are the fixed pts of $U(1)$.

Want to get this from susy localization. Take $E=\Sigma \mathcal{F}$

(In gen. for $E \to M$ a v. bundle, $\mathcal{F}$ supermult $\mathcal{F}_E$)

$\mathcal{F}^{\omega}(\mathcal{F}_E) = \mathcal{F}^{\omega}(M, \Lambda^*E)$

Concrete: In local coord. patch on $M$, $E=\mathcal{T}_H$, even coords $x^i, i=1,\ldots,2n$

Odd $\tilde{w}$, $\tilde{w} \cdot dx_i$

$$f = \tilde{w} + f_i(x) \tilde{w}^i + f_{ij}(x) \tilde{w}^i \tilde{w}^j$$

$$\frac{\partial}{\partial x} + f_i(x) \frac{\partial}{\partial \tilde{w}^i} + f_{ij}(x) \frac{\partial}{\partial \tilde{w}^j} \tilde{w}^i$$

Take action:

$$S = -i\int (H+\omega) = -i \int (H + \omega_{ij} \tilde{w}^i \tilde{w}^j)$$

$$Z = \int d\tilde{w} e^{-S}$$

Canonical measure on $\tilde{w}$ $= \frac{1}{\text{Vol}(\mathcal{F})}$
\[ 0 \rightarrow \pi^\ast TM \rightarrow T\Sigma \rightarrow TM \rightarrow 0 \Rightarrow \exists \pi T\Sigma = \exists \pi TM \oplus \exists \pi TM \oplus \text{trivial} \]

If we focus on tensors first:
\[ \xi = (i \alpha)^n \int_\mu \frac{\omega^n}{n!} \quad \text{vol} \]

Want to compute by localization. \( S \) is inv. under the odd \( \alpha \).
\[ Q = d + z \gamma = \gamma \partial_x + z \partial_t \]

\[ \frac{1}{2} \left[ Q, Q \right] = \left[ d + z \gamma \right]^2 = \int_y = p \left( \frac{1}{2} \gamma \partial_t \right) \delta_y + \gamma \partial_t \delta_x. \]

\[ Q \xi = (d + z \gamma) (H + \omega) = dH + z \gamma \omega = 0. \]

Want to get localization to fixed pts \( q \) by perturbation \( S \rightarrow S + \lambda Q \varphi \), \( \lambda \rightarrow 0. \)

For this: Fix a \( U(1) \)-inv. metric \( g \) on \( M \). Take \( \varphi = g(y) = g_{ij} \psi_i \psi_j = \psi_1 \psi_i. \)

\[ Q \varphi = g(\gamma(y)) - d(g(y)) \]

\[ Q^2 \varphi = 0 \quad \text{def. by def.} \]

Now take \( S \rightarrow S + \lambda Q \varphi \), as before, \( \xi \) is independent \( \lambda \) (uses \( Q^2 \varphi = 0 \)).

Take \( \lambda \rightarrow 0 \) use steepest descent around fixed pts of \( \gamma \):

\[ \sum_{x \in H; \gamma(x) = 0} \left( \text{vol} \right)^n \frac{g(y)}{1!} \left( \frac{d(g(y))(x)}{\sqrt{\det g(y)}(x)} \right) (x, x) . \]

Num/def. are both valued in \( \Lambda^\top T^*_{x_c} H \). To calculate this ratio:

\[ \text{Consider local model situation - diagonalize } \ U(1) \text{-action on } T_{x_c} H \quad \left( \begin{array}{c} \cos(k_i x) \\
\sin(k_i x) \end{array} \right) \text{acts on } \mathbb{R}^2 \quad \text{by} \]

\[ \left( \begin{array}{cc} \cos(k_i x) & \sin(k_i x) \\
-\sin(k_i x) & \cos(k_i x) \end{array} \right) . \]

\[ E.g., \quad \text{take } T_{x_c} H = \mathbb{R}^2, \quad U(1) W^t = k \]

\[ g = \text{std metric}, \quad g = dr^2 + r^2 d\theta^2 \]

\[ \omega = \text{sympl. form}, \quad \omega = r dr \wedge d\theta \]

\[ \gamma = k^2 \theta, \quad h = \frac{1}{2} k^2 \theta, \quad g(\gamma, \gamma) = k^2 r^2 \]

\[ d(g(y)) = 2k^2 r^2 dr \wedge d\theta \quad \sqrt{\det g(y)} = 2k^2 r^2 dr \wedge d\theta \]
\[
\int e^p = \sum_{x < \frac{1}{\varepsilon}} \frac{\varepsilon e(\varepsilon x)}{\pi i k_i(x)} = \frac{2 \alpha}{\pi} (1 + \alpha \varepsilon)^{\frac{1}{2}}
\]
Choose $\text{Riem. 1-mfd } (X, \gamma)$. X apt: either $X = [0, T]$ or $X = S^1(T)$, param. $X$ by $t$ ("fine"). $L_X$ will be 1-dim space of "masses on $x"$.

Data:
- A Riem. mfd $(X, g)$ "target"
- $V : \gamma \to \mathbb{R}$ "potential"
- $\gamma \subset \gamma$.

For $x = S^1$:
$$L_{S^1} \gamma = \left\{ \phi : S^1 \to \gamma \right\}.$$

For $x = [0, T]$:
$$L_{[0, T]} \gamma = \left\{ \phi : [0, T] \to \gamma \mid \phi(0) = y_0, \phi(T) = y_1 \right\}.$$

Define:
$$S(\phi) = \int_X d\text{vol}_X \frac{1}{2} \left( g(\phi' , \phi') + V(\phi) - \frac{1}{2} R(\phi) \right).$$

Scalar curvature of $\gamma$

$$= \int_X d\phi \left[ \frac{1}{2} g_{i\bar{j}} (\phi(h)) \phi_i' \phi_{\bar{j}}' (h) \phi_i (h) \phi_{\bar{j}} (h) + V(\phi(h)) - \frac{1}{2} R(\phi(h)) \right].$$

As $g_{i\bar{j}} = 1$

$$= \int_X \left( \frac{1}{2} g_{i\bar{j}} (\phi(h)) \phi_i' \phi_{\bar{j}}' + V(\phi) - \frac{1}{2} R(\phi) \right) dh.$$  

We want to consider
$$Z : S^1(h) = \int_{S^1(h)} d\phi e^{-S(\phi)} \approx \text{Cologues) measure exists.}$$

$$Z \int_{L_{[0, T]} \gamma} e^{-S(\phi)} = \int_{L_{[0, T]} \gamma} d\phi e^{-S(\phi)}.$$  

**Discretization:**
- Say $x = [0, T]_{y_0}$: Replace $X$ by lattice: $N_{\Delta T, N}^T \gamma_{y_0}$.
  - Discretized field space $\mathcal{Z}_{X : \mathbb{N}} := \{ \text{piecewise geodesic paths } \phi : X \to \mathbb{N} \}$ which only two at the $t_i$. 

$\text{Pos. temperature } \gamma \in \mathfrak{h}, \mathfrak{H}$.
Restrict $S$ to the paths, get $s: \mathbb{C}; N \to \mathbb{R}$

$C_N \subseteq \mathbb{C}^{N+1}$ = $s$ has $\pi$-measure $d\mu_N = \frac{1}{(4\pi \Delta t)^{\frac{N}{2}}} \prod_{k=1}^{N-1} d\text{vol}_y (\phi_{\Delta t})$

Define: $Z_N = D x^N = \int e^{-s} d\mu_N$ try to take limit as $N \to \infty$.

Then: limit exists, $k_N$ heat kernel $k_N(x_0,y_0)$

Def.: Fix $y$. For $t \to \infty$, the heat kernel $(defined by \gamma)$ is a smooth fun on $\mathbb{R}^N$ obeying

$\frac{d}{dt} k_t(x,y) + \Delta k_t(x,y) k_t(x,y) = 0$

$\lim_{t \to 0^+} k_t(x,y) = \delta(x,y)$

Ex.: Show that on $Y = \mathbb{R}^n$ when $V = 0$, the heat kernel is $k_e(x,y) = (4\pi t)^{-\frac{n}{2}} \exp \left( -\frac{1}{4t} ||x-y||^2 \right)$

The heat kernel is a kernel for the operator $k_t$ of heat evolution for time $t$ i.e.,

$\text{evolving solutions } u(x,t) \text{ of } \left[ \partial_t + (-\Delta + V(x)) \right] u(x,t) = 0$

evolving solutions $u(x,t)$ of $\left[ \partial_t + (-\Delta + V(x)) \right] u(x,t) = 0$

forward in time $u_t$ is a smoothing operator: maps distributions $\to C^\infty$ for $t > 0$ gives a linear operator

on $L^2(U), u_t = e^{-t(-\Delta + V)}$.

Then: $\lim_{N \to \infty} Z_N = k_N(x_0,y_0)$

Hausdorff Phi: (when $V = 0$):

$U(t) = (U(t)^N \text{ for } (U(t)^N)$

$k_N(x_0, y_0) = \int_{\mathbb{C}^{N+1}} \frac{N-1}{\pi} dx_0 \frac{N-1}{\pi} dy_0 \int_{\mathbb{R}^N} \delta_{\Delta t}(x_0, y_0) \text{ (ie)}$

In short time asympt. of $k_N(x, y) \sim (4\pi \Delta t)^{-\frac{N-1}{2}} \exp \left( \frac{1}{4t} \int \frac{d(x,y)^2}{\Delta t} \right)$

Subsitute: \begin{align*}
&\int_{\mathbb{C}^{N+1}} \frac{N-1}{\pi} dx_0 \frac{N-1}{\pi} dy_0 \int_{\mathbb{R}^N} \delta_{\Delta t}(x_0, y_0) \exp \left( \frac{1}{4t} \int \frac{d(x,y)^2}{\Delta t} \right) \\
&= \int_{\mathbb{C}^{N+1}} \frac{N-1}{\pi} dx_0 \frac{N-1}{\pi} dy_0 \exp \left( -\frac{1}{4t} \left( \frac{d(x_0, y_0)}{\Delta t} \right)^2 \right) \\
&= Z_N
\end{align*}
So far, 1-d QFT

\[ S(\phi) = \int_Y \left( \frac{1}{2} g(\Phi, \Phi) + V(\Phi) - \frac{1}{3} R(\Phi) \right) \text{dvol}_Y \]

path integral in this theory compute heat flow, i.e.

\[ \int \mathcal{T} e^{-\mathcal{H}T} \]

\[ \mathcal{H} = -\frac{1}{3} \Delta + V \]

\[ \mathcal{T} \left[ e^{-\mathcal{H}T}, \mathcal{H} \right] \]

\[ \phi(x) \mapsto \phi^\oplus(x) = \int_Y k_T(x, y) \phi(y) \text{dvol}_Y \]

\[ \frac{\partial}{\partial T} (e^{-\mathcal{H}T} \phi) = -\mathcal{H} e^{-\mathcal{H}T} \phi. \]

The harm. Osc.

\[ Y = \mathbb{R}, \quad V(x) = \frac{1}{2} \omega^2 x^2, \quad \omega \in \mathbb{R} \]

\[ \mathcal{H} = -\frac{1}{2} \Delta + \frac{1}{2} \omega^2 x^2 : \quad L^2(\mathbb{R}) \xrightarrow{\text{2nd Order}} L^2(\mathbb{R}) \]

**Eigen.s:**

\[ \phi_0(x) = e^{-\omega x^2/2} \]

\[ \phi_1(x) = x e^{-\omega x^2/2} \]

\[ \phi_2(x) = (x^2 - \frac{1}{4\omega}) e^{-\omega x^2/2} \]

\[ \phi_n(x) = \frac{1}{\sqrt{n!}} \omega^{\frac{1}{4}} x^n e^{-\omega x^2/2} \]

They obey \( \mathcal{H} \phi_n = (n + \frac{1}{2}) \omega \phi_n \)
Thus,
\[ Z_{S'}(T) = \text{Tr} \, e^{-HT} = \sum_{n=0}^{\infty} \exp \left( -\omega (n + \frac{1}{2}) T \right) = \frac{1}{2 \sinh \left( \frac{\omega T}{2} \right)} \]

Sigma model into \( S' \) (particle on a circle)

\[ Y = S'(R), \quad V = 0. \]
\[ H = -\frac{1}{2} \Delta^R \quad \text{on} \quad H = L^2(S'(R)) \quad \text{and} \quad S'(R) = R/R. \mathbb{Z} \]

Basis of eigenfunctions:
\[ \varphi_0(x) = 1, \quad \varphi_{2n-1}(x) = \sin \left( \frac{2\pi n x}{R} \right), \quad \varphi_{2n}(x) = \cos \left( \frac{2\pi n x}{R} \right) \]

Eigenvalues of \( H \):
\[ 0, \quad \frac{4\pi^2 n^2}{R^2} (\text{multiplicity } 2) \quad \text{for } n \neq 1. \]

So,
\[ Z_{S'(T)} = \text{Tr} \, e^{-TH} = 1 + 2 \sum_{n=1}^{\infty} \exp \left( -\frac{4\pi^2 n^2 T}{R^2} \right) \]
\[ = \sum_{n=-\infty}^{\infty} \exp \left( -\frac{4\pi^2 n^2 T}{R^2} \right) = \varphi^2 \left( \frac{4\pi^2 T}{R^2}, z = 0 \right) \quad (\text{Jacobi-\( \phi \))} \]

Ex.: \[ Y = S'(R) \]
\[ S_\kappa : H \to H, \varphi \mapsto \varphi (\cdot + \kappa) \]
\[ [S_\kappa, H] = 0. \]
Compute \[ \text{Tr}_H (e^{-TH} S_\kappa) \] to get \( \psi(0, \frac{\kappa}{R}) \).
Remark: This QFT has 'global symmetry' \( G = U(1) \).

We can formulate it on a Riem. mfd \( X \) equipped w/ \( G \)-bundle w/connection.

In this example, this means taking \( X = S^1 \) w/ flat \( G \)-bundle determined by its holonomy \( \kappa \in U(1) \).

('twisted diry cond. ' for loops: \( \phi(x + \tau) = \phi(x) + \kappa \frac{\tau}{\tau} \mod \pi \))

**Determinants**

Recall: \( V \) f.d. vs., \( M : V \oplus V \rightarrow \mathbb{R} \), \( dp \) measure on \( V \), \( c \in \mathbb{R} \)

\[
\int_{\mathbb{R}^c} \left( \frac{1}{2} \dim V \right) dp e^{-\frac{1}{2} M(x, x)} = \frac{dp}{\sqrt{\det(cM)}} \tag{\ast}
\]

Note: \( \sqrt{\det H} \) is naturally a density on \( V \) (under change of coords, \( H \rightarrow A^T H A \)

\( \det H = (\det A)^2 \det H \) )

\( \text{Dens}(V) = \Lambda^\text{top}(V^*) \otimes \text{Or}(V) \).

Our discretized path integral's have form of LHS of (\( \ast \)) (because \( S = \int g(\phi, \phi) + V(\phi) \)

(if we take \( c = \Delta t = \frac{1}{N} \)). Limit as \( N \rightarrow \infty \) of LHS exists

Limit as \( N \rightarrow \infty \) of RHS exists

How to interpret \( N \rightarrow \infty \)-limit of RHS ? \( (\dim V = N) \)

In f.d. case, choosing a metric on \( V \) w/ \( \|dp\| = 1 \). \( \text{Let us identify} \ V \cong V^* \)

\( \text{Mat}: V \oplus V \rightarrow \mathbb{R} \)

\( \Delta: V \rightarrow V \)

\[
\int_{\mathbb{R}^c} \frac{dp}{\sqrt{\det(M \Delta^2)}} = \frac{1}{\sqrt{\det(A)}}
\]
\[ Y = A, \quad V = \frac{1}{2} \omega^2 x^2 \]

\[
\begin{pmatrix}
\sum_{n=1}^{N-1} a_n^2 & \frac{1}{2} \omega^2 T & -1 & 0 & -1 \\
\frac{1}{2} \omega^2 T & \sum_{n=1}^{N-1} b_n^2 & -1 & 0 \\
-1 & 0 & -1 \\
0 & -1 & \cdots & \cdots & \cdots
\end{pmatrix}
\]

\[
\lim_{N \to \infty} \frac{1}{\text{det } A} = \frac{1}{2 \sinh(\frac{\omega T}{2})}
\]

**Bilinear Form for Discre. Action**

**1-dim. Version**

\[
V = \left\{ \phi : s^1 \to \mathbb{R}, \quad \int_{0}^{1} \phi^2 \text{ finite} \right\}
\]

\[
S = \frac{1}{2} \int dl \left( \frac{d}{dt} \left( \phi \phi^* + \omega^2 \phi \phi^* \right) \right)
\]

\[
\phi(t) = c \sqrt{T} + \sum_{n=1}^{\infty} \frac{\sqrt{\frac{T}{2\pi}}} n \left( \alpha_n \sin \left( \frac{2\pi n t}{T} \right) + \beta_n \cos \left( \frac{2\pi n t}{T} \right) \right)
\]

Take the norm \[ ||\phi||^2 = c^2 + \sum_{n=1}^{\infty} (\alpha_n^2 + \beta_n^2) \quad \text{[Ludwig]} \]

\[
S(\phi) = \frac{1}{2} \left( \omega^2 T c^2 + \sum_{n=1}^{\infty} \left( 1 + \frac{\omega^2 T^2}{4 \pi^2 n^2} \right) (\alpha_n^2 + \beta_n^2) \right)
\]

Eigenvalues of corresponding operator:

\[
\lambda = \omega^2 T, \quad 1 + \frac{1}{4 \pi^2 n^2} \quad \text{(mult. 2)} \quad \forall n > 0
\]

\[
\sqrt{\text{det } A} = \omega T \prod_{n=1}^{\infty} \left( 1 + \frac{\omega^2 T^2}{4 \pi^2 n^2} \right) = 2 \sinh \left( \frac{\omega T}{2} \right).
\]
Last time: \( d-A \neq T \Rightarrow Z = \{ X \rightarrow Y \} \)

\[ S = \int \frac{d\phi}{\sqrt{g}} (\frac{1}{2} g_{\mu \nu} \frac{\partial \phi}{\partial x^\mu} \frac{\partial \phi}{\partial x^\nu} + V(\phi) - \frac{1}{6} \lambda^2) \]

\[ Z \leftarrow \text{heat flow on } Y \]

\[ \int e^{-s} \]

Two ways to think about \( Z_{\phi}(T) \)

1) as \( \int \) over loops in \( Y \) in case of harm. osc. \( Y = \mathbb{R}^2, V = \frac{1}{2} \omega \cdot \omega \cdot x^2 \)

2) as \( \text{Tr}_n e^{-HT} \), \( H = \mathcal{L}^2(Y) \)

\[ \sum_{\lambda} e^{-\tau \lambda}, \lambda = \text{eigenvalues of } H \]

In case \( Y = \mathbb{S}^1(\mathbb{R}) \)

2) gave \( Z_{\phi}(T) = g^\phi (x = \frac{\text{Re} e^{i T}}{\tau}, z = 0) \)

1) gave \( Z = g^\phi (x = \frac{i \tau e^{i T}}{\text{Re}}, z = 0) \)

Symmetries

Recall odd case: Vector fields on \( T \) which annihilate \( S \) (or \( G\mathcal{C} \) preserving \( S \))
give constraints on corr. fcn.

\( \langle 0 \rangle = \langle 0^g \rangle \)

\( \langle x \rangle = 0 \quad x \in g \)
Symmetries of our model

1) isometries of $X$

2) isometries of $Y$ (preserving $V$)

For $X=S^1(T)$, $t$ is $\text{Isom}(X) = U(1)$ $t \rightarrow t + c$

$\rightarrow \langle \sigma_1(t_1) \sigma_2(t_2) \ldots \sigma_n(t_n) \rangle = \langle \sigma_1(t_1+c) \sigma_2(t_2+c) \ldots \sigma_n(t_n+c) \rangle$

$\sigma$ means a local observable, i.e., $\sigma \in C_x \rightarrow \mathbb{R}$, depending only on finite

jet of $\phi$ at $t$.

$\phi$ and its derivatives

Similar formula for $g \in \text{Isom}(Y, V)$

Notation: to describe the action of $\text{Isom}(Y) = U(1)$ on $g$, just write action

of Lie algebra

$t \rightarrow t + \varepsilon$,

then $\phi(t) - \phi(t + \varepsilon) = \phi(t) + \varepsilon \phi'(t)$, i.e., $\delta \phi = \varepsilon \phi'$

Effective field theory 1-dimension

Consider a system $Y = \mathbb{R}^2_{x,y}$, $V = \frac{1}{2} x^2 + \frac{1}{2} \omega^2 y^2 + \frac{1}{4} x^2 y^2$

Intuition: If $\omega \ll 1$, should be able to eliminate $y$ — yes, why? (Born-Oppenheimer)

 homeowners
Suppose we're interested in normalized expectation values involving only \( x \), e.g.,

\[
\frac{\langle x(0) x(t) \rangle_{S^1}}{Z_{S^1(t)}} = \frac{1}{Z_{S^1(t)}} \int_{S^1(t)} x(0) x(t) \, dx \, dy \, e^{-S(x,y)}
\]

An asymptotic series in \( \mu \) can be evaluated by Feynman diagrams, as we did in O-d:

Recall in O-d had

\[
\begin{array}{c}
\text{vertices} \\
\lambda_{ij} \rightarrow \lambda_{ij} x_i x_j x_k x_l
\end{array}
\]

Basic ingredient: Green's fns: \( D_x \), \( D_y \) on \( S^1 \)

\[
(\partial_x^2 - \mu^2) D_x(t) = \delta(t) \quad D_x(t) = \frac{1}{2} \sum_{n \in \mathbb{Z}} e^{-it + nT}
\]

\[
(\partial_y^2 - \omega^2) D_y(t) = \delta(t) \quad D_y(t) = \frac{1}{2\omega} \sum_{n \in \mathbb{Z}} e^{-\omega t + nT}
\]

Then

\[
\frac{\langle x(0) x(t) \rangle_{S^1}}{Z_{S^1(t)}} = \int_0^t \int_0^t \int_0^t + \int_0^t + \int_0^t + \int_0^t + \int_0^t + \int_0^t
\]

\[
D_x(t) + \frac{\mu}{2} \int_{S^1(t)} dt' D_x(t') D_x(t-t') D_y(0) + \ldots
\]
How about effective field theory? Want to "integrate out" $y(t)$?

Formally,

$$S_{\text{eff}}(x) = -\log \int dy e^{-S(x,y)}$$

$$S_{\text{eff}}(x) = \int_t^t + \int_t^t + \int_t^t + \cdots$$

$$= \int dt \left[ \frac{1}{2} x(t)^2 + \frac{1}{2} \frac{\mu}{\epsilon} \Delta_y(0) x(t)^2 \right]$$

$$= \int dt \left[ x(t)^2 + V(x(t)) \right] \text{ as it involves } \int dt \int dt'$$

**But:** $D_y(t-t')$ decays exponentially, away from $t=t'$ diagonal

- To expand the non-local interaction:

$$\int dt \int dt' x(t)^2 x(t') D_y(t-t') = \int dt \int dt' D_y(t-t') \left[ x(t)^2 + 2 x(t)^2 x(t-t') + \right.$$

Now do the $\int$ over $t'$ to get:

$$\int dt \left[ x(t)^4 + \frac{\epsilon_2}{\omega} \left( x(t)^2 \frac{\epsilon}{\omega} \frac{1}{2} x(t)^3 \right) + \cdots \right]$$

Get an infinite series of "higher-derivative interactions" suppressed by powers of $\omega$.

So, for process dominated by paths where these extra terms are small gives a systematic expansions in powers of $\frac{1}{\omega}$. For $\omega \to 0$ then theory looks local after all.
Supersymmetric QM

Y Riem. multif or 1d QFT had $C = \operatorname{Hop}(x,y)$.
Now: add fermions! Make a new QFT w/ $C = \prod T \operatorname{Hop}(x,y)$.
A point of $C$ means
\[
\cdot \phi : x \rightarrow y \quad \text{actions} \quad X \in [p,T]_{x_0} \quad \text{or} \quad S'(T)
\]
\[
\cdot \psi \in \prod \Gamma(\phi^* T y)
\]

Rem.: $C = \prod T H$ for any $H$ has canonical measure (if $H$ oriented)

Action: $S \in C^0(C)$,
\[
S(\phi, \psi) = \int dt \frac{1}{2} \left[ g(\phi, \dot{\phi}) + g(\psi, \nabla \psi) \right]
\]
$\nabla \psi$ = pulled-back Levi-Civita.

Rem.: Second term is like $\int dt \, \psi \dot{\psi}$.
For bosons $\int dt \, \dot{\phi} \phi = -\int \dot{\phi}^2 dt \Rightarrow$ it's zero
Here
\[
\int dt \, \dot{\psi} \psi = -\int dt \, \psi \dot{\psi} = \int dt \, \psi \psi
\]
$S$ is inv. under time translation
\[
\delta \phi = \varepsilon \phi, \quad \delta \psi = \varepsilon \nabla \phi
\]
But it has another odd symmetry:
\[
\delta \phi = \varepsilon \phi \quad (\varepsilon \text{ odd})
\]
\[
\delta \psi = -\varepsilon \phi
\]
$Q = \int dt \left( \varphi(t) \frac{\partial}{\partial \varphi(t)} - \dot{\varphi}(t) \frac{\partial}{\partial \dot{\varphi}(t)} \right)$

$H = \int dt \; \dot{\varphi}(t) \frac{\partial}{\partial \dot{\varphi}(t)}$

10-03-17

Perturbation theory in QM

(En, FT→QH)

S over path w/ S Gaussian → ν-dice det.

Non-Gaussian → can make asymptotic approx in the non-Gaussianity

\textbf{Ex.} "quadratic oscillator" \hspace{1cm} \begin{array}{c}
\uparrow \\
\downarrow \\
X \rightarrow Y
\end{array}

$Y = \tilde{Y}, V = \frac{1}{2} \lambda y^2 + \frac{\lambda}{4\gamma} x^4$

$H = -\frac{1}{2} \delta_x^2 + V(x)$

Want: Eigenvalues of $H$ on $L^2(\mathbb{R})$

Extract them from $Z_{S'(T)} = \sum_n e^{-E_n T}$ \hspace{1cm} $H | \psi_n \rangle = E_n \psi_n$

Feynman rules:

$Z_0 = 2 (\lambda=0)$

$\frac{Z_{S'(T)}}{Z_0}$ \hspace{1cm} is sum over diagrams

Rules: \begin{align*}
H & : V \mapsto \mathbb{R}, C : V \mapsto \mathbb{R} \\
H^* & \in \mathbb{C}
\end{align*}

$\mathbb{C} \in (\mathbb{C}^*)^I$
Here we have
\[ V = \left\{ uS \rightarrow \mathbb{R} \right\}, \]
\[ H(t,x) = \int_0^T e^{-\frac{1}{2}(\frac{1}{\omega}(\omega t + \frac{1}{2}\omega^2 x^2))} \, dt. \]

Its "inverse" is
\[ G(t,s) = G(t-t') = \frac{1}{2\omega} \sum_{n \in \mathbb{Z}} e^{-\omega t', t+nT}. \]

This "inverts" \( H \) in the sense that
\[ H(G(t'), t') = f(t'), \]

Recall in [ed.]: \( H^{-1}(\eta, u) \rightarrow v \)
\[ H^{-1}(v) = \Omega \]

\[ \log \left( \frac{Z}{Z_0} \right) \approx -\frac{\lambda}{2} T \log \left( \frac{\omega}{T} \right)^2 + O(\lambda^2) = \frac{\lambda}{32\omega^2} (\coth \frac{\omega T}{2})^2 + O(\lambda^2) \]

\[ \log Z = \log \left( \sum_k e^{-\frac{\lambda}{2} T} \right), \quad \text{as} \quad T \to \infty \quad \log Z \sim -T E_0(\lambda) \quad \left( \log \left( \frac{Z}{Z_0} \right) \sim -T E_0(\lambda - \lambda_0) \right) \]

So,
\[ E_0(\lambda - \lambda_0) \sim \lambda \lim_{T \to \infty} \frac{1}{32\omega^2} (\coth \frac{\omega T}{2})^2 = -\frac{\lambda}{32\omega^2}. \]
We want to write down a 1-d QFT w/ \( \mathcal{Z} = \prod T \mathcal{H}_o P (x, y) \).

"Points" of \( \mathcal{Z}_X \):
\[ \phi : x \to y \]
\[ \psi \in \prod T (\phi \star T y) \]

**Action \( S = S_o (\mathcal{Z}) \):**
\[
S(\phi, \psi) = \int dt \left\{ \frac{i}{2} \left( \psi \dot{\phi} \dot{\phi} + \frac{1}{2} \left( \partial_i \dot{x} \dot{\phi} + \frac{1}{2} \right) \right) \right\}
\]

**Symmetries**: time translation \( \delta \phi = \epsilon \dot{\phi}, \delta \psi = \epsilon \dot{\psi} \)

\[ (i.e., \text{ in any local coord. on } y: \delta \phi = \epsilon \dot{x}, \delta \psi = \epsilon \dot{y}) \]

Also odd symmetry: \( \delta \phi = \epsilon \dot{\phi}, \delta \psi = -\epsilon \dot{\psi} \)

**Claim**: \( QS = 0 \).

\( e.g. \) if \( y = \mathbb{R}^n \), \( S = \frac{1}{2} \left( \dot{\phi} \dot{\phi} + \dot{\psi} \dot{\psi} \right) \)

\[
\delta S = \frac{1}{2} \left( 2 \epsilon \dot{\psi} \dot{\phi} \dot{\phi} - \epsilon \dot{\phi} \dot{\phi} \dot{\phi} \right) = 0
\]

**Ex.** \( \frac{1}{2} [Q, Q] = H \).

"\( Q \) gives a \( \Gamma \) of time translations"
Ex: Show that if we vary the metric $g$ on $Y$, $g_1 \rightarrow g_1 + \delta g$, then the action $S$ varies by

$$S \rightarrow S + \partial \mathcal{F}, \quad \mathcal{F} = \int dt \left( \frac{1}{2} (\delta g)_{ij} \partial_i \phi \partial_j \phi \right)$$

So: Finally, expect $Z$ is independent of metric on $Y$.

$Z$ is independent of metric on $Y$ but depends on a spin structure on $Y$!

**Spinors**

**Def:** $V$ a f.d. vector space: $T(V)$ tensor algebra is $T(V) = \bigoplus_{n=0}^{\infty} V^\otimes n$

With product $(V^\otimes \cdots \otimes V^k) \times (V^l \otimes \cdots \otimes V^j) := (v_1 \otimes \cdots \otimes v_k) \otimes (v_l \otimes \cdots \otimes v_j)$

**Def:** $V$ a f.d. real v.s. or pos.-def. quadratic form $\langle \cdot, \cdot \rangle$. Then

$$\text{Cliff}(V) := T(V)/\text{2-sided ideal generated by } V \otimes V + \langle v, v \rangle,$$

i.e. "impose the relation $v^2 = \langle v, v \rangle 2 I$ $\mathbb{Z}_2$-graded."

$$\frac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1) = -\langle v_1, v_2 \rangle$$ in $\text{Cliff}(V)$

and if $V$ has basis $e_1, \ldots, e_n$, then $\text{Cliff}(V)$ has $\mathbb{R}$-basis

$$1, \quad e_i, \quad e_i e_j, \quad e_i e_j e_k, \quad e_i e_j e_k e_l$$

$$\dim \text{Cliff}(V) = 2^n.$$
Rem.: $\text{Cliff}(V)$ is a deformation of $\Lambda^*(V)$

Def.: $\text{Pin}(V)$ is the group of all elements $v, v_1, v_2 \in \text{Cliff}(V)$, $(v_i, v_i) = 1, v_i \in V$.

Def.: $\text{Spin}(V) = \text{Pin}(V) \cap \text{Cliff}^o(V)$, even part

Fact: $\text{Spin}(V)$ acts on $V$ by

$$\text{Cliff}(V) \ni v \mapsto gvg^{-1} \in \text{Cliff}(V),$$

$g \in \text{Spin}(V)$

This action preserves the inner product $\langle \cdot, \cdot \rangle$ on $V$. So, get map $\text{Spin}(V) \to \text{SO}(V)$. This map is a double cover.

Notation: $\text{Cliff}(n) = \text{Cliff}(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ generated by $e_i e_j = \frac{1}{2} \delta_{ij} - \delta_{ij}$.

Ex.: $\text{Cliff}(1) \cong \mathbb{C}, \text{Cliff}(2) \cong \mathbb{R}, \text{Spin}(1) \cong \mathbb{Z}/2\mathbb{Z}.$

$\text{Cliff}(2) \cong \mathbb{H}, \text{Cliff}^o(2) \cong \mathbb{C}, \text{Spin}(2) \cong \text{SU}(2)$

$\text{Cliff}(3) \cong \mathbb{H} \otimes \mathbb{H}, \text{Cliff}^o(3) \cong \mathbb{H}, \text{Spin}(3) \cong \text{SU}(2)$

Ref.: Nishida, "Spin." eq.

Def.: Fix oriented Riem. $\mathbb{M}^n$, then have $P \to Y$ bundle of orthonormal frames, principal $\text{SO}(n)$-bundle $\text{Spin}(n) \to \text{SO}(n)$.

A spin structure is a lift of $P$ to a $\text{Spin}(n)$-bundle.

$\big(Q \rightarrow \text{Spin}(n)\big) \downarrow \downarrow \downarrow \big(P \rightarrow \text{SO}(n)\big)$

Ex.: $Y = S^n$ with metric orientation. $P$ is $\text{SO}(n)$-bundle, i.e., $P = Y$

$
\rightarrow$ a spin structure on $Y$ is a double cover of $Y$

There are up to $\infty$.  

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Last time: Cliff alg.

\[ V \text{ real vector space} \rightarrow \text{Cliff}(V) \text{ associated algebra, } \mathbb{Z}/2 \text{ graded} \]

\[ \text{w/ pos. def. } \langle, \rangle \]

\[ \Pi_n(V) \]

\[ U \]

\[ \text{Spin}(V) \rightarrow \text{SO}(V) \text{ double cover} \]

\[ \text{Spin}(V) = \text{Pin}(V) \cap \text{Cliff}^+(V) \]

\[ \text{Spin}(V) \rightarrow \text{SO}(V) \text{ double cover} \]

If \( V = \mathbb{R}^n \), have Spin\(n\), Cliff\(n\), etc.

Remark:

\[ \Pi_n(\text{SO}(n)) = \begin{cases} 
 1 & n = 1 \\
 2 & n = 2 \\
 4 & n = 3 \\
 8 & n \geq 4
\end{cases} \]

So, for \( n \geq 3 \), Spin\(n\) is universal cover of \( \text{SO}(n) \).

Spin structure: \( M \) oriented, Riem. \( FM \) bundle of frames principal \( \text{SO}(n) \)-bundle.

Spin structure is lift of \( FM \) to spin-bundle.

On \( M = S^3 \), 72 spin st. up to \( \mathbb{Z} \), \( Q \mapsto Q \)

Exercise: If \( M \) admits spin structures ("spinable"), shows that the spin structure on \( M \) up to \( \mathbb{Z} \) form a torsor for \( H^1(M, \mathbb{Z}/2) \).

(Idea: Given a spin structure \( Q \), and a double cover \( C \), \( Q \otimes_{\mathbb{Z}/2} C \) twist \( Q \) by \( C \) another spin structure.)
$H = \mathbb{C}P^2$ does not admit a spin structure. Why?

Look at hyperplane $H = \mathbb{C}P^1 \subset \mathbb{C}P^2$.

$T\mathbb{C}P^2$ restricted to $H$ is $\cong \mathcal{O}(x^2) \quad \text{from } N\mathbb{C}P^1$ from $T\mathbb{C}P^1$

Over $H \cong \mathbb{C}P^1$, $\mathcal{O}(x^2)$ is given by

\[
\begin{pmatrix}
\text{rotation of } x^2 \\
\text{rotation by } \theta
\end{pmatrix}
\]

\[S' \longrightarrow \text{SO}(4), \quad \pi_1(\text{SO}(4)) \cong \mathbb{Z}/2\mathbb{Z}
\]

and this loop represents the non-trivial element $\Rightarrow$ doesn't lift to $\text{Spin}(4)$.

**Remark:** Every opt orient. 2-fld is spinable (by Euler class $c = d - g$ is even)

Every opt orient. 3-fld is spinable (by TH is trivial!)

A opt simply conn. opt. 4-fld is spinable

$\Rightarrow$ intersection pairing $H^2(\mathbb{H}, \mathbb{Z})$ is even.

**Prop.** $\text{Cliff}(2n)$ has a $\mathbb{Z}$ $\mathbb{Z}/2\mathbb{Z}$-graded rep. (irreducible rep.)

$S = S^0 \oplus S'$

\[\dim_{\mathbb{C}} S^0 = \dim_{\mathbb{C}} S' = 2^{n-1}\]

Spin(2n) acts on $S^0, S'$. Up to $\cong$ and $S^0 \cong S'$. $S$ is the unique rep of $\text{Cliff}(2n)$. 

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E.g. \( \mathbb{C}l(\mathbb{H}) \) has \( S = S^0 \oplus S^1 \)

\[
\begin{align*}
e_1 & \rightarrow (0, i) \quad e_2 \rightarrow (0, i) \quad e_3 \rightarrow (0, i) \\
e_4 &= -e_2 e_1 \\
e_5 &= e_2 e_1
\end{align*}
\]

**Def.** Fix a spin manifold \( M \) (\( M \) odd w/ spin structure \( \Omega \) \( \rightarrow \) \( T M \rightarrow M \) )

The associated bundle \( SM = \Omega \times_{\text{Spin}(n)} S \) is called spinor bundle.

- \( SM \) has connection \( \nabla \) induced from Levi-Civita
- \( SM \) has action \( \varphi : T H \rightarrow \text{End}(SM) \) induced from action of \( \text{Cl}(n) \) on \( \mathbf{R}^n \)

\( \gamma \) (Dirac operator)

\( D : C^\infty(SM) \rightarrow C^\infty(SM) \) given by \( D = \sum_{i=1}^n e_i \gamma_i \delta_i \nabla \), where the \( \{ e_i \} \) form an \( \mathbf{R} \)-vector basis for \( TM \).

**Ex.** If \( H = \mathbf{R}^2 \), then \( SM \) is trivial bundle \( \mathbf{C}^2 \) and

\[
\begin{align*}
\gamma &= e_1 \delta_1 + e_2 \delta_2 = \begin{pmatrix} 0 & i \delta_2 \\ -i \delta_2 & 0 \end{pmatrix} \\
\nD^2 &= \begin{pmatrix} -\delta_2 & -\delta_1 \\ 0 & -\delta_2 \end{pmatrix} = -\Delta, \\
\text{def.} \quad \Delta : C^\infty(SM) \rightarrow C^\infty(SM)
\end{align*}
\]

**Def.** \( M^{2n} \) be a spin manifold. Define

\[
\Delta = -D^2 \quad \text{spin Laplacian},
\]

on even degree forms.

\[
\Delta : C^\infty(SM) \rightarrow C^\infty(SM)
\]
\[ C_x = \Pi T \text{Hop}(x, y) \]

\[ \phi : x \to y \]

\[ \psi \in \Pi T^1(\phi^* Ty) \]

Add virtual field \( Q \) on \( C_x \)

\[ \Delta \]

\[ \frac{1}{2} [Q, Q] = H. \]

Let's try to calculate \( Z_x = \int_{C_x} e^{-S(\psi, \phi)} \) by discretization.

Define \( C_{x, N} = \{ \text{piecewise geodesic paths } s_i \to y \} \)

\( + \\text{offsets } \psi_i \in T_{\phi_s(1)}(y) \)

Make discretized action see \( \Gamma_{x, N} \)

\[ Z_{x, N} = \int e^{-S_{x, N}} \]

We'll use formula \( \nabla_{\phi} = \partial \phi + \dot{\phi} \) relative to trivialization of \( \phi^* Ty \).

Claim: If we fix a trivial \( F_o \) of \( \phi^* Ty \) and use it for the discretization, then the limit of the functional as \( N \to 0 \) exists, but it depends on \( F_o \) as follows.
For each fixed $\phi : S^1 \rightarrow Y$, the space of trivializations $F_\phi$ of $\phi^*TY$ is a torsor for $
abla^S(SO(\mathfrak{su}))$.

The four torsions

$$W(F_\phi) = -W(F_\phi^T)$$

(10)

Accepting (10), we have a problem in defining $Z_\phi$.

Need some extra structure on $Y$ which picks out a "good" class of fibers $F_\phi$.

Fix a spin structure on $Y$, then we can choose only those $F_\phi$ which lift

to the spin structure.  --- cure the sign problem

\[ \begin{align*}
\phi^*Q & \rightarrow \text{Spin}(\mathfrak{su}) \\
\phi^*F_\phi & \rightarrow SO(\mathfrak{su})
\end{align*} \]

A toy model

Fix $Y$, dim $Y = 2$, $\phi : S^1(T) \rightarrow Y$, trivialize $\phi^*TY$ as $SO(\mathfrak{su})$-bundle over $S^1$ of $Y$.

\[ \nabla_L = \partial_L + \alpha R \]

$R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\text{Re} R \rightarrow$ gauge transformation

Discretized action:

\[ S = \frac{1}{2} \sum_{i=1}^{N} \left( \frac{1}{q_i} - \frac{1}{q_{i+1}} + \frac{q_i^2 - q_{i+1}^2}{N^2} \right) \]

\[ W = \sum_{\text{d, p, o, d}} e^{-S} \]

Claim: \[ \lim_{N \rightarrow \infty} W = \frac{1}{2} \sin \left( \frac{\pi T}{2} \right) \]
So far: Supersymmetric QFT: Theory of "supermaps"

\[ e_x = \left\{ \begin{array}{cc} \cdot \phi : X \to Y & \\
\cdot \psi \in \Gamma (\pi^* T Y) & \end{array} \right\} Y \text{ Riem. spin} \]

Last time: discussed integrating over fermions \( \phi \).

Brief digression: A slight extension of the story of bosonic QFT we told earlier.

Data: Y Riem. manifold

\( V: Y \to \mathbb{R} \)

\( E \) Out-vector bundle over \( Y \) w/ metric and complex case. \( \nabla \).

Thus, we formulate "heat equation for sections of \( E \)" \( f_t \in \Gamma (Y, E) \)

\[ \frac{\partial f_t}{\partial t} + Hf_t = 0 \quad H = -\frac{1}{\varepsilon} \Delta + V \quad \Delta = \nabla^* \nabla \]

"Heat kernel coupled to \( E \)" is

\[ k_t \in \Gamma (E^* \otimes E) \]

\[ \begin{array}{c}
E^* \otimes E \downarrow \\
Y \times Y
\end{array} \]

Like one \( (g_0, y_1) \) is \( E^*_{g_0} \otimes E_{g_1} = \text{Hom}(E_{g_0}, E_{g_1}) \).

Can define 1-d QFT coupled to \( E \):

\[ Z = \text{Hap}(X, Y) \]

as before

\[ Z_{\text{S}^1(T)} = \int \text{Tr} \text{Hap}_\phi e^{-S(\phi)} \]

\[ \text{diff. species of particles} \]
Prop.: \[ Z_{S'}(T) = \text{Tr} \mathcal{L}(E) e^{-HT} \]

discr., take limit, which is well-def. 

In our SUSY QH example, \( S \) over fermions will produce an effective theory of maps \( X \to Y \) coupled to the bundle \( E = S', \) and \( S \).

More exactly, we'll get

\[ Z_{S'}(T) = Z(S') - Z(S) \]

\[ e^L(E) = Z S(T) \text{ for theory coupled to } E. \]

\[ \rightarrow Z_{S'}(T) = \text{Tr} \mathcal{L}(E') e^{-HT} - \text{Tr} \mathcal{L}(E) e^{-HT} = S \text{Tr} \mathcal{L}(E) e^{-HT} \]

\[ \forall \alpha \]

Prop.: \( S \text{Tr} e^{-HT} \]

\[ = \dim \ker \phi^0 - \dim \ker \phi^1, \]

\[ \phi = \begin{pmatrix} 0 & \phi^1 \\ \phi^0 & 0 \end{pmatrix}, \quad \phi^* = \phi \]

\[ = \text{ind} \phi^0 = \dim \ker \phi^0 - \dim \text{coker} \phi^0. \]

Rem.: \( \phi \) is a 1st order elliptic diff. op. Elliptic means that the principal symbol is invertible - e.g., for \( \Delta \) on \( \mathbb{R}^n \),

\[ \Delta = -\sum_i \partial^2_{x_i} + \text{(lower order)} \text{ in local coords} \]

symbol replace \( \partial^2_{x_i} \text{ with } \sum k_i^2 \)

Diff. op. \( \mathcal{D} : E \to \mathcal{F} \text{ over } X \)

\[ \mathcal{D} \in \Gamma(\pi^* \mathcal{Hom}(E, E)), \pi : T^* X \to X. \]

Ellipticity: \( \mathcal{D} \) is invertible off zero section.
Fact: For Del Pezzo over $\mathbb{C}$, $\text{ker } D$ are fin. dim. consist of $\mathcal{C}^0$ sections.

Proof of Prop: Unitary rep. theory of the algebra $[\mathcal{F}, \mathcal{F}] = \Delta$ ($\mathcal{F}, \Delta$ s.a.)

Decompose into irreps.

$\Delta$ is central $\Rightarrow$ can diagonalize; multiple of 1 in each rep.

Fix an irrep where $\Delta = E$, $E \in \mathbb{R}$.

Claim: $\mathbb{Z}_2$-graded irreps $V$ are of three types:

1) $\dim V = 1|1|$, $E > 0 \leftarrow$ states $\Phi \in V^0$
2) $\dim V = 1|0|$, $E = 0$
3) $\dim V = 0|1|$, $E = 0 \leftarrow$ one state $\Psi$, $\Phi \Psi = 0$.

$$
\begin{array}{ccc}
E & V^0 & V^1 \\
\downarrow & \downarrow & \downarrow \\
\Phi & \Phi & \Phi \\
\end{array}
$$

all cancel out in $\text{Str}_V e^{-TH}$.

Rem.: If $\Delta = -E_{\geq 0}$ and ker $\Delta = \ker \Phi$

Proof:

$-E \langle \Phi, \Phi \rangle < \langle \Psi, \Delta \Psi \rangle = <\Psi, -\Phi \Phi > = -<\Phi, \Phi > 0$.

Proof of Claim:

(i) $\text{Str}_V e^T \Delta = e^T e \text{ Tr } e^{-T} e$

(ii) $\text{Str}_V e^T \Delta = \text{Str}_V 1 = 1$

So: $Z_{\mathcal{F}(T)} = \text{ind } \mathcal{F}$

Next aim: use this to give a new formula for $\text{ind } \mathcal{F}$ (index theory)
Fix $X$.

Recall: Given a symmetric function $\mathcal{C}(\{y_i\})$, define a char. class of $\text{SO}(n)$-bundles $\Xi$ over $X$: Fix a connection $E$. Locally block-diagonalize

$$ F = \bigoplus_i \begin{pmatrix} 0 & F_i \\ -F_i & 0 \end{pmatrix}, \quad F_i \in \Omega^2(X). $$

Then take the form $C(\{F_i\}) \in \Omega^*(X)$.

Ex.: $X$ opt. Riem. w/f.

$$ p_i(X) \in \Omega^i(X), \quad p_1(X) = 1 + p_2(X) + p_3(X) + \ldots, \quad p_k \in \Omega^{4k}(X) $$

is the class associated to $C = \frac{1}{2\pi i} (1 + g_i^2)$ (Pontryagin class).

Ex.: $p_1(X) = -\frac{1}{4\pi^2} \text{Tr} F \wedge F$.

Recall: $p_k(X)$ is the rank rep. of a class in $H^{4k}(X, \mathbb{Z})$.

Ex.: \( \int_{\mathbb{F}} p_1(S^4) = 0, \quad \int_{\mathbb{CP}^2} p_1(\mathbb{CP}^2) = 3, \quad \int_{\mathbb{CP}^2} p_1(H^3) = -48 \)

(Recall: this $\Rightarrow \mathbb{CP}^2$ and $H^3$ are chiral, not diffeo to $\overline{\mathbb{CP}^2}$ or $\overline{H^3}$)

Recall: If $\dim X = 4$, $p_1(X) = 3 \text{ sign } (X)$.

Def.: $X$ opt. Riem. w/f.: $\hat{A}(X) \in \Omega^*(X)$ is char. class of $\mathcal{T}X$ assoc. to

$$ \mathcal{T} \begin{pmatrix} y_i^2/2 \\ \sinh(y_i^{2}/2) \end{pmatrix} = \mathcal{T} \begin{pmatrix} 1 - \frac{y_i^2}{24} + \frac{7y_i^4}{5760} + \ldots \end{pmatrix} $$

$$ \hat{A}(X) = 1 - \frac{1}{24} p_1(X) + \frac{7p_2(X)^2}{5760} - \ldots $$

degeen = $0$.
Then (Atiyah-Singer)

\[ \text{Ind } \phi_0 = \int \hat{\Theta}(x). \]

Only pick up parts of right order, dep. on dim(X).

E.g., if dim X = 4, then

\[ \text{Ind } \phi_0 = -\frac{\int \chi(X)}{24}. \]

E.g.: 
- \( S^4 \): Ind \( \phi_0 = 0 \)
- \( K^3 \): Ind \( \phi_0 = 1 \)
- \( \mathbb{C}P^3 \): not spin!

Then (Rehren)

\( X \) spin 4-wwd = \( 16! \) sign(\( \chi \)).

Where we are:

1-d SUSY QFT (in SUSY QM)

\( \mathcal{H} = \frac{1}{2} \) or \( \mathcal{H} = f \)

\( \mathcal{Q} = \mathbb{H} \)

Different flavors of QM: all theories of maps \( X \to Y \). All of them have formal

diagram = Riem.

Structure of 1-d QFT:

Hilbert space \( \mathbb{H} \) (assoc. to a point)

Hamiltonian: \( H: \mathbb{H} \to \mathbb{H} (e^{-iH} \text{ assoc. to } \mathbb{H}) \)

At 12 2017
Example:

<table>
<thead>
<tr>
<th>$N$</th>
<th>$H$</th>
<th>$H$</th>
<th>$Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$L^2(Y)$</td>
<td>$-\frac{1}{2}\Delta$, $\Delta = \text{Laplace-Beltrami}$</td>
<td>$-$</td>
</tr>
<tr>
<td>1</td>
<td>$L^2(Y, SY)$</td>
<td>$-\frac{1}{2}\Delta$, $\Delta = \text{spinor Lapl.}=\Phi^2$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>2</td>
<td>$L^2(Y, \mathfrak{A}^T\mathfrak{A})$</td>
<td>$-\frac{1}{2}\Delta$, $\Delta = [d, d^<em>)$ $= dd^</em> + d^* d$</td>
<td>$\mathfrak{a}, d^*$</td>
</tr>
<tr>
<td></td>
<td>$\Omega^*_L(Y)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$Y$ Kähler $\Rightarrow 4$

$Y$ Hyperkähler $\Rightarrow 8$

\[ H_k ; g \text{ Kähler wgt. 2 diff. $C^\infty$ str. $I^2 = I = I^2$ $= -1$, $I_\mathbb{C} = -IT, H^2 = I$} \]

\[ \mathcal{L}_X = \{ (\phi, \psi) | \phi: Y \to Y, \psi \in \Omega_T(\phi^* T Y) \}, \mathcal{Z}_{S(T)} = \{ d\phi, d\psi e^{-\mathcal{L}(\phi, \psi)} \} \]

Claim: \[ \int \text{ over } \psi \text{ leaves for fixed } \phi: \mathcal{Z}(T) \to Y \]

\[ \omega(\phi) = e^{-\mathcal{L}(\phi)} \int\int S(T) H_1 [\phi^* T Y] \]

Remark: Not easy to define the $\mathcal{Z}$ over fermions by discretization! ("Fermion doubling problem") Instead, we approach via no-thru determinants.

\[ \mathcal{S}(\phi, \psi) = \int dt \left[ \frac{1}{2} \mathcal{L}(\phi, \phi) + \frac{1}{2} \mathcal{L}(\psi, \psi) \right] \]

Want \[ \int d\psi e^{-\frac{1}{2} \mathcal{S}(\phi, \psi)} \], i.e., Pfaffian of skew pairing $(\phi_1, \psi) \to \frac{1}{2} \mathcal{S}(\phi_1, \phi_2)$
First, what's the determinant?

(To make sense out of this, we'll need a metric on $\Gamma(p^*(\gamma))$.

Look at the operator $\nabla_t$: Eigenvalues of $\nabla_t$ are $\frac{d\omega_i}{T}(k + \frac{\omega_i}{2\pi})$, $k \in \mathbb{Z}$, $e^{\pm i\omega_i}$ are eigenvalues of $\text{Hol}(p^*(\gamma))$. (To see this, choose local trivializations $\nabla_t = \partial_t + \left( \begin{array}{c c}
\frac{2i\omega_i}{T} & -\frac{i\omega_i}{T} \\
-\frac{i\omega_i}{T} & -\frac{2i\omega_i}{T}
\end{array} \right)$

If we take the Fourier modes to be an ON basis, we would get

\[ \det(\text{pairing}) = \prod_{k \in \mathbb{Z}} \prod_{i=1}^{n} \frac{2i\omega_i}{T}(k + \frac{\omega_i}{2\pi})(k - \frac{\omega_i}{2\pi}). \]

What if we use Sobolev norm

\[ \|p\|^2 = \text{det} (g_t(V_t, V_t) + g_t(V_t, V_{\xi t}))^2 \]

Then, norm of k-th Fourier mode $\|\cdot\|_k^2$, still orthogonal. So, the det. relative to this norm is for large $k$

\[ n \prod_{k \in \mathbb{Z}} \prod_{i=1}^{n} \frac{2i\omega_i}{T}(1 + \frac{\omega_i}{2\pi})(1 - \frac{\omega_i}{2\pi}) \rightarrow \text{convergent} \]

\[ \rightarrow \text{det is an honest fun of } \omega_i \in \mathbb{C} \text{ on double zeros at } \omega_i = 2\pi k. \text{ Periodic under } \omega_i \rightarrow \omega_i + 2\pi \]

\[ = 1 \text{ (up to a const.) it's } \left( \frac{2\sin(\frac{\omega_i}{2})}{\omega_i} \right)^2 \]

\[ \text{hope: real for } \omega_i \in \mathbb{R} \text{ (this is the only possibility)} \]
Now what about the Pfaffian?

**Obvious Candidate:** \(\prod_{i=1}^{n} i \sin \left( \frac{\pi i}{2} \right)\)

Now depends on \(\kappa\) mod \((4\pi)\) -> mod spin structure.

**Refs:** Witten "global anomalies in string theory."
Atiyah "circular symmetry in stationary phase approx."

**Lemma:** If \(A \in \text{SO}(n)\) has eigenvalues \(e^{\pm i\alpha_i}\), then
\[
\prod_{i=1}^{n} \left( i \sin \left( \frac{\pi i}{2} \right) \right)^2 = \det \left( 1 - A \right).
\]

**Prop:** If \(A \in \text{Spin}(2n)\), then
\[
\det \left( 1 - A \right) = \det \left( \left( ST_5 A \right)^2 \right).
\]

\[\varepsilon : \text{spin}(2n) \to \text{so}(2n) = \det(1 - \varepsilon(A))\text{ spinor rep.}\]

**Proof:** For \(n = 1\),
\[
\varepsilon(\alpha) = \begin{pmatrix}
\cos \alpha & \sin \alpha \\
-\sin \alpha & \cos \alpha
\end{pmatrix}.
\]

Recall spin rep of \(\text{spin}(2)\):
\[
\dim S = 1
\]
\[
ST_5 \theta = e^{i\pi/4} - e^{-i\pi/4} = 2i \sin \left( \frac{\pi}{2} \right).
\]

\(\Box\)
\[ Z = \int_F e^{H+\omega} \text{Euler}(NF) \] (ABBV formula)

This led to

\[ \text{F} = \text{fixed locus} \]
\[ \text{H}^0 \text{ of } U(1) \]

Our case now:

\[ C = \pi_1 T(Y^2) \]
\[ H = \mathcal{L}Y \text{ (loop space)} \]
\[ U(1) \text{ action rotates the loops} \]
\[ \text{gauge}: H(\phi) = \frac{1}{2} \int g(\psi, \bar{\psi}) \]
\[ \omega(\gamma) = \frac{1}{2} \int g(\psi, \bar{\psi} \gamma) \]

What is \( F^2 \) space of constant loops: \( F^2 Y \in \mathcal{L}Y \).

\[ \text{ind } \phi^0 = 2 = \int_y \frac{1}{\text{Euler}(N+1)} \]
What is $N^F$?

$$T_0 \otimes Y = \text{Hap}(S^1, T_0 Y) \cong T_0 Y \oplus \bigoplus_{k \geq 1} (T_0 Y \otimes T_0 Y).$$

$N^F \cong \bigoplus (T_0 Y \otimes T_0 Y)$, $U(1)$ acts on $k$th summand w/ weight $k$.

Curvature of $N^F$, induced from curvature $\rho$ of $TY$.

Oct 17 '17

SUSY QM: super theory of maps: $\phi: X \to Y$

$$\downarrow$$

(SUSY+QFT)

Hamiltonian language:

$$Z_{S(\Gamma)}^{n} = \text{ind } \phi^n \quad \phi: L^{2}(Y, S) \to L^{2}(Y, S)$$

"$S_{\Gamma} e^{-T\Gamma}$"

Path $\int$ language:

$$Z_{S(\Gamma)}^{n} = \int dx \, dp \, e^{-\frac{S}{\hbar^n}} \left\{ \frac{1}{\text{Euler}(N^F)} \right\}$$

where

$$\bigcup_{U(1) \text{ relating loops}}$$

fixed locus = $\{ \text{ fixed loops } \}$

$a_{\mu}^Y$

$$N^{F} \cong \bigoplus_{k \geq 0} (T_0 Y \otimes \mathbb{R}^2)$$

$a$ $\text{def} y$ is just a map $S \to T_0 Y$.
Recall: \[ \text{generator of } \mathcal{U}(1) \text{-action} \]
\[ E \overset{\mathcal{U}(1)}{\longrightarrow} \mathfrak{so}(E)^* \]
\[ \text{In our case: } E = N^\mu; \text{ curvature is } R \otimes 1 = \left[ \bigoplus_{i=1}^n R_i (0 \otimes 1) \right] \]
\[ T_Y \otimes \bigoplus_{k > 0} R^k \]
\[ U(1) \text{-action is } \left[ \bigoplus_{k > 0} k (0 \otimes 1) \right] \]

Exercise: On \[ \mathbb{R}^k \otimes \mathbb{R}^2 \]
show that
\[ \mathcal{P} \left( a \left[ (0 \otimes 1) \otimes 4 \right] + b \left[ (0 \otimes 1) \right] \right) = a^2 - b^2 \quad (= a(b + a - b)) \]

Formally then,
\[ \text{Euler}(N^\mu) = \prod_{i=1}^n \prod_{k > 0} \left( k + R_i \right) \left( k - R_i \right). \]

Try regulating it, as we did before: Then Euler \( (N^\mu) = \prod_{i=1}^n \sin(\pi R_i) \).

Euler \( (N^\mu) \) is almost \( \hat{A}(Y)^{-1} \), not quite: its k-form part differs by rescaling by \( (4\pi)^k \).

More precise treatment should fix even the constant.

Q: Compute the example in each dim. to fix it.

\[ \sum_{k} (\hat{A}(Y)) \rightarrow \text{index theory for } \hat{A}. \]

Rem.: I many extensions/harps of ASIT. One simple one: fix another metric \( M^E \) w/ connection over \( Y \). Then I twisted Dirac op.

\[ E_E : S \otimes E \rightarrow S \otimes E \]
AS IT for this says

$$\text{ind } \hat{A} = \int \hat{A}(y) \cdot \text{ch}(E)$$

$\text{ch}(E)$ is char. class of $\mathcal{U}(w)$-bundles associated to $P(\mathbb{C}^1) = \Sigma \mathbb{C}^2$.

E.g., $\text{ch}(E) = r_k(E) + c_1(E) + \frac{1}{2}(c_1^2 - 2c_2) +$. 

This variant can also be "proved" using SUSY QM - now coupled to $E$.

$$Z_{\text{SUSY}} = \int d\phi \, d\psi \, \text{Tr} \, \hat{A}(\phi) e^{-S(\phi, \psi)}$$

SUSY completion of usual Høl

Friedman–Willey

$$e^{i\theta \partial + q \partial_x}$$

In SUSY localization for this SUSY QM, we'll get

$$\int e^{H + \omega \int Y E^a(N \Phi)}$$

and now $\omega = ?$, so get $\int \hat{A}(Y)^{-1}$

For application to Gauss–Bonnet, signature, etc., use this variant (e.g., G–B comes from setting $E = S$)

Another important ext” : families index then - given a family of (twisted) Dirac ops, (parameter space $B$) define a super-vector bundle ("index bundle") over $B$.
Index bundle carries a natural connection, can ask about its holonomy, curvature...

[Biswut–Freed...]

In QM, this means we have a family of 1-d SUSY QFTs, i.e., family of SUSY QM. Such a family always gives a vector bundle of "ground states" over parameter space, w/ connection ("Berry phase").

[Ahrens–Winkel... I give SUSY explanation of families index than using Berry connection.

Next we're going to d-dim. QFT, $X =$ surface. One special case: $X = T^2$; then formally $Z_x$ is supervision of $\mathbb{C}^d \otimes Y \to$ might expect index than for Dirac op. on $\mathbb{C}^d \otimes Y$ (need spin structure on $\mathbb{C}^d \otimes Y$!). This $3!$; elliptic genus.

The free boson in 2 dimensions

2-d sigma model: fix data:

- $X$: Riem. 2-mfld
- $Y$: Riem. mfld
- $V: Y \to \mathbb{R}$

$Z = \text{Hap}(X, Y)$

$d\varphi: T_x X \to T_{\varphi(x)} Y$

Ex: Say $X = S^1 \times S^1$, $Y = \mathbb{R}$, $V = 0$. Then can Fourier expand $\varphi$ in modes:

$\varphi(x, t) = \sum_{n \in \mathbb{Z}} a_n(t) e^{i 2\pi n x / L}$, $\overline{a_n} = a_{-n}$.

Now,

$S(\varphi) = \int dt \left[ \frac{1}{2} \dot{a}_0(t)^2 + \sum_{n \neq 0} \left( |\dot{a}_n(t)|^2 + \frac{4\pi^2 n^2}{L^2} |a_n(t)|^2 \right) \right]$. L
This defines an action for an infinite set of functions \( a_n(t), t \in S^1, a_n : S^1 \to \mathbb{R} \)
\[
S(\vec{a}^n) = \sum_{n > 0} S_n(\vec{a}^n),
\]
each \( S_n \) separately is action of harmonic oscillator w/ potential \( V(x) = \frac{1}{2}x^2 \).
What should we get for \( Z_{S_n} \)? (canonical approach)
Naive guess: \( Z = \prod_{n \in \mathbb{Z}} e^{-T H_n} \), where \( H = \bigotimes_{n \in \mathbb{Z}} H_n, \quad H_n = \sum_n H_n \)
\( H = H_0 \otimes 1 \otimes 1 \otimes \ldots + 1 \otimes H \otimes 1 \otimes \ldots \)
What are the eigenvalues of \( H \)? Start w/ lowest one (ground state).
Should be \( \Sigma \) of ground state energies in each \( H_n \). That would give
\[
E = \sum_n E_n = \sum_{n > 0} \frac{\omega_n}{2} \frac{2\pi n}{L} \sum_{n > 0} n.
\]
How to make sense of this \( \Sigma \)?

Oct 19, 2017

---

Last time: first look at 2-d QFT.

"Free boson": theory of maps \( \phi : x \to \mathbb{R}, S(\phi) = \int dx \left( \frac{1}{2} \phi' \phi + \frac{1}{2} \phi^2 \right) \)

Case \( \phi = T^x = s'(t), s' = 2 \).

Two routes: 1) \( Z_X = \det \) (quadratic pairing \( \omega_1 \omega_2 \to \int \langle \phi_1 \phi_2 \rangle \))

2) \( Z_X = T_H e^{-T H} \)

By Fourier expanding \( \phi(x) = \sum_n a_n e^{\frac{2\pi in}{L}} \)
\[
H_L = \bigotimes_{n > 0} H_n = \bigotimes_{n > 0} L^2(\mathbb{R})^{\otimes 2}
\]
\[
H_0 = -\frac{1}{2} \frac{\partial^2}{\partial x^2}, \quad H_n = -\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{2} \omega_n^2 x_n^2, \quad \omega_n = \frac{2\pi n}{L}
\]
Rem: $H(L)$ has no countable basis. Naively basis really would be

$$\otimes_{k \in \mathbb{N}} \mathbb{R}\phi_k \otimes \mathbb{R}\phi_k$$

Recall for 1d QFT $\phi: \mathbb{R} \to \mathbb{C}$, $H=\hat{\mathcal{L}}(\phi)$  
\[ \forall \phi \in \mathcal{H} \text{ is like an initial condition for } \phi. \]

It has to do with initial cond. for $\phi$ at fixed $t$, say $t=0$.

i.e., it has a basis consisting of states.

$$|\phi_0, \phi_1, \phi_2, \ldots \rangle = \otimes_{k \in \mathbb{N}} |\phi_k \rangle \otimes \otimes_{k \in \mathbb{N}} |\phi_k \rangle$$

More generally, given an $n$-dim QFT “defined” by path $f$ and $(n-1)$-MF $M$, want to define $H_M^*$. Heuristic: “path-$f$ quantization” - look at space of extrema of $S$ when theory is on $M \times \mathbb{R}$. (E.g., for 1-d QFT, $\mathcal{S} = \int \text{param. geodesics in } Y \times T Y$)

Then, $\mathcal{S}_M$ carries a symplectic structure. E.g., use $TY \simeq T^* Y$.

Then, $H_M$ is the quantization of $(\mathcal{S}_M, \omega)$.

![Diagram](image)

$$X \mapsto \mathcal{Z}_X$$

$$H_{d-1} \mapsto H^*_{d-1} = \text{ quant. of } \mathcal{S}_M \text{ on } H_{d-1} \text{ (fields on } H_{d-1} \times \mathbb{R})$$

$$\mathcal{Z}_X \in \text{Hom} \left( H_M \otimes H_2 \circ \text{Hom} H_2 \right)$$

![Diagram](image)

$$\mathcal{Z}_X \circ \mathcal{Z}_X$$

diff. Hamiltonians
\[ Z_x = \frac{1}{\text{Tr}_H} e^{-H^T}. \]  What's lowest eigenvalue of \( H \)?

Formally \( E = \sum_n E_n = 2 \sum_{n>0} \frac{\omega_n}{\omega} = \frac{4\pi}{L} \sum_n n \).

If we defined the theory by discretization, effect should be to remove high-\( n \) part while leaving small \( n \) part unchanged.

In our ansatz, we'll replace the contribution \( E_n \) by \( E_n f(E_n) \), \( f(E) \) cutoff for \( \varepsilon \), lattice spacing.

Take \( f(E) = e^{-\varepsilon E / \varepsilon} \). Then what happens to our sum?

\[
E = \frac{2\pi}{L} \sum_{n>0} n e^{-\frac{n\varepsilon}{L}} = \frac{4\pi}{L} \frac{(-1)}{d\varepsilon} \sum_n e^{-\frac{n\varepsilon}{L}}
\]

\[ = \frac{2\pi}{L} \frac{d}{d\varepsilon} \frac{e^{-\varepsilon / L}}{1 - e^{-\varepsilon / L}} = \frac{4\pi}{L} \frac{e^{-\varepsilon / L}}{(e^{\varepsilon / L} - 1)^2}
\]

Now expand around \( \varepsilon = 0 \):

\[ E(\varepsilon) = \frac{4\pi L}{\varepsilon^2} - \frac{4\pi}{\varepsilon L} + \ldots
\]

The divergent part as \( \varepsilon \to 0 \) is prop. to \( L \).

\[ \Rightarrow \text{we could absorb it by adding to the action } S = \frac{2\pi}{\varepsilon^2} \text{ and } S = \int \frac{2\pi}{\varepsilon^2} + \| \phi \|^2
\]

After making this addition, get \( E = -\frac{\pi}{\varepsilon L} \), finite.

The rest of the spectrum of \( H \) is "easy": ground state was \( \Psi_0 \otimes \Psi_0 \) \[ E_0 = -\frac{\pi}{\varepsilon L} \]

\[ \Psi_m \otimes \Psi_n \quad E \sim \frac{\pi}{n \varepsilon L} + \Sigma n \cdot m 
\]

Only fin. many \( n, m \)
Altogether, this gives \( q = e^{-\frac{2\pi i}{L}} \), \( \mathcal{Z}(q) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{N}(g)_n (-1)^n \)

\[
\mathcal{Z}_X = \frac{V}{2\pi i \gamma} \mathcal{N}(g)_0^{-2}
\]

(need to deal w/ the zero mode \( \alpha_0 \), \( \mathcal{H}_0 = L^2(\mathbb{R}), \mathcal{H}_0 = -\partial_x^2 \))

Plan:

\[
\mathcal{Z} = \text{Map} \left( S^1 \times S^1, \mathbb{R} \right) \bigg| \text{This has the symmetry } y \rightarrow y + c \text{ (R-action)}
\]

\[
\mathcal{Z} = \int \phi \phi^* \bigg| \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{N}(g)_n (-1)^n \bigg| dt \rightarrow \mathcal{Z} \text{ is infinite}
\]

Deal w/ it by regulating \( L^2(\mathbb{R}) \rightarrow L^2 \left( S^1 \times S^1 \right) \)

Modular property of \( \mathcal{Z}(q) \)

\[
q = e \quad 2\pi i \tau, \quad \tau \equiv \frac{\tau}{L}
\]

\( \mathcal{Z}_X \) is invariant under \( \tau \leftrightarrow \tau' \)

All this could be generalized to \( X \) a tiled torus

\[
X = \mathbb{C}/\left( \mathbb{Z} \oplus \mathbb{Z} \tau \right) \quad \tau \in \mathbb{H}
\]

\[
\mathcal{Z}_X = \frac{V}{2\pi i \sqrt{\text{Im} \tau}} \left| \mathcal{N}(g)_0 \right|^{-2}
\]

What we learned: 2-d QFT is a factory of modular forms.

E.g. Witten genus of \( Y \rightarrow \) comes from \( \mathcal{N} = (0,1)^n \) SUSY \( \sigma \)-model in \( d=2 \).
Last time: Free bosons in 2 dimensions.

Theory of maps $\phi : X \to \mathbb{R}$

We calculated for $X = S^1 \times S^1$

$$Z_X = \frac{\nu}{\nu^1} q(q)^{-2}$$

$$q = e^{-2\pi \nu \zeta}$$

$$y(q) = \sum_{n=1}^{\infty} \frac{\mu}{\nu} (1-q^n)$$

V came from need to regulate an "IR divergence"—came from translation action $\phi \to \phi + c$.

One way to do this: instead of computing $Z_X$, take instead

$$\left< e^{-\varepsilon (\phi(x))^2} \right>, p \in X$$

and look at limit $\varepsilon \to 0$.

**NB:** This ansatz ($Z_X$) is invariant under $T \to Z_1$, i.e., it's ten on moduli space of flat (rectangular) tori.

Fix tilded torus

$$Z = \frac{\nu}{\nu_1} |y(q)|^{-2}$$

$$\nu \to \nu_1, \ z \to -\frac{1}{\varepsilon}, \ X = \mathbb{C}/\mathbb{Z} + \mathbb{Z} \mathbb{Z}.$$ 

Remark: This only depends on the conformal class of the metric on $X$.

In fact, the free boson in 2 dim is an example of a conformal field theory.

To see why, let's consider making a rescaling of the metric on $X$.

$$S(\phi) = \int_{S^1} d\theta d\phi \nu \phi \delta_{\phi} \phi \delta_{\phi} \phi$$
If we rescale \( q \rightarrow \lambda q \)
\[
\sqrt{\det q} \rightarrow \lambda \sqrt{\det q}
\]
\[
(q'^{-1})^{ij} \rightarrow \lambda^{-1} (q'^{-1})^{ij}
\]
\[
(\phi \rightarrow \phi \text{ in } \mathbb{R})
\]
so this is invariant! (even if \( \lambda=\lambda(x) \text{ coordinate})

\[
\Rightarrow Z^1 \text{ is scale invariant and moreover, } \langle \phi(x), \phi(x_n) \rangle \text{ also scale invariant.}
\]

\[
X \rightarrow Z = \frac{1}{2\pi i} q \frac{d}{dq} \quad (q = e^{2\pi i \theta})
\]

How about maps \( X \rightarrow Y \), \( Y = S^1 / G(\text{Ker}) \), \( X = T^2 = S^1(\mathbb{L}) \times S^1(\mathbb{T}) \)?

In path language, \( \mathcal{Z} = \text{Hap}(X, Y) \text{ is disconnected.} \)

\[
\mathcal{Z} = \bigcup_{(n_1, n_2) \in \mathbb{Z}^2} \mathcal{Z}_{n_1, n_2} \quad \phi(x+L, t) = \phi(x, t) + 2\pi n_1 \mathbb{L} + 2\pi i n_2 \mathbb{T}
\]

So path \( \int \rightarrow \sum_{n_1, n_2} n_1 \mathbb{L} + 2\pi i n_2 \mathbb{T} \). In Hamiltonian formalism, the space \( S \) of classical solutions on \( S^1(\mathbb{L}) \times \mathbb{R} \) is decomposed \( S = \bigcup_{n \in \mathbb{Z}} S_n \), and each \( S_n \) is acted on by the group \( \text{Isom}(Y) \cong S^1 \)

\[
\Rightarrow \text{roughly speaking, } S_n = S^1 \times \mathbb{A}_{n, \text{reduced}}
\]

\[
\Rightarrow \text{in quantization, get a factor } L^2(S^1).
\]

\[
\Rightarrow \text{Fourier expansion in } L^2(S^1) \text{ gives } H^m \cong \bigoplus_{m \in \mathbb{Z}} H^m_{n_1, n_2}
\]

\[
\text{lowest eigenvalue of } H \text{ in } H^m_{n_1, n_2}:
\]

\[
E_{n_1, n_2} = \left( \frac{m}{\mathbb{L}} \right)^2 + \left( \frac{m_2}{\mathbb{T}} \right)^2 \quad \text{(maybe up to factor } \frac{1}{2})
\]

\[
\text{from winding around space } \quad \omega_m = \frac{2\pi m}{\mathbb{L}}
\]

\[
\text{from momentum } \quad \omega_{n_2} = \frac{2\pi m_2}{\mathbb{T}}
\]

\[
L^2(S^1) \text{ w/ } H = \frac{d^2}{dt^2}
\]

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If look at
\[ \phi(R) = x \cdot n \cdot d\phi \cdot \frac{R}{\Sigma} \]
\[ \sum_{L(S)} \int \Delta \phi \cdot \frac{L}{\Sigma} \]
\[ \sum_{\Sigma} \frac{1}{q} \left( \frac{m_n}{n} - \gamma (n, R) \right)^2 - \frac{1}{q} \left( \frac{m_n}{n} - \gamma (n, R) \right)^2 , \gamma = c^{-\frac{2\pi R}{\Sigma}} \]

Invariant under \( R \rightarrow \frac{1}{R} \) ! This theory can't tell the difference between \( S(R) \), \( S\left(\frac{1}{R}\right) \)! i.e., here we have one QFT with two different classical descriptions!
Not saying that the theory w/ fixed \( R \) has a special symmetry. 

For pencil:
The duality extends also to correlation fns,
E.g., \( \forall x \in X \) have a map \( "d\phi \cdot \omega" : \Sigma \rightarrow T_x^* X \) (in coords, look at \( x, y, \partial x \).
This gives a \( T_x^* X \)-valued observable. Similarly, have
\[ "\# d\phi (x)^* : \Sigma \rightarrow T_x^* X \]
Hodge star on \( X \)
And:
\[ \langle d\phi (x), d\phi (x) \rangle \omega / Y = S(R) \]
\[ \langle \# d\phi (x), \# d\phi (x) \rangle \omega / Y = S\left(\frac{1}{R}\right) \] "T-duality"

(How to motivate this?)

Rem: \( d\phi \) always \( dC^2 = 0 \), i.e., \( dC^2 \ldots = 0 \)
trivial kinematic relation
\( (C^2 = \# d\phi \) also obeys \( dC^2 = 0 \), \( d\# d\phi = 0 \).)
A SUSY analogue of the theory of maps $T^2 \to Y$ will give:

Recall $\hat{A}$-genus, characteristic class assoc. to sym. $p^n$

$$
\prod_{i} \frac{2i\pi}{\sinh(2i\pi)} \quad \text{this came from 1-d SUSY QFT}
$$

\underline{Witten genus:} Consider the form

$$
\prod_{i} \left[ \frac{2i\pi}{\sinh(2i\pi)} \right] \left[ \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{(1-q^n e^{x}) (1-q^n e^{-x})} \right]^{\frac{1}{2}}
$$

of $q$. Expanding in powers of $q$, get series of char. classes

$$W(q) = \sum_{n=0}^{\infty} q^n W_n(q).$$

Evaluated on some particular $Y$, get a $q$-series

$$W_Y(q) = \sum_{n} a_n(Y) q^n.$$

If $Y$ is spin, all coeff. $a_n(Y) \in \mathbb{Z}$.

\textbf{Then:} If $\rho_Y(Y) = 0$, then $W_Y(q)$ is modular.
Last time: 2d free boson on $S^1$, theory of maps $X \rightarrow S^1(\mathbb{R})$.

To study partition func correctly, on $X = S^1(T) \times S^1(\mathbb{R})$.

For maps to $\mathbb{R}$, expand

$$\phi(x,t) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} a_n(t) e^{2\pi i n x / l}.$$ 

For maps to $S^1$, modify in 2 ways:

1) add a term for "winding":

$$\phi(x,t) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} a_n(t) e^{2\pi i n x / l} + a(t) x / l, \quad \text{we Z}$$

$$\Rightarrow \phi(x+L,t) = \phi(x,t) + 2\pi.$$ 

2) $a(t)$ is now a map $\mathbb{R} \rightarrow S^1(R\mathbb{L})$

The action $S$, written in terms of $a_n(t)$, is exactly as it was for maps $T \rightarrow \mathbb{R}$, except for an extra term

$$\int_{S^1} \left( \frac{\partial a(t)}{\partial t} \right)^2 = \frac{\omega^2}{L^2}$$

$$\Rightarrow \text{in } S^1 \times S^1 \text{ contr. from all fields } a_n(t), \omega \neq 0 \text{ are just as before: } \eta(x) \rightarrow$$

winding produces

$$\sum_{n \in \mathbb{Z}} e^{-\omega^2 n^2 / L^2}.$$ 

The field $a(t)$ has action $\int dt \frac{1}{2} \dot{a}(t)^2$ and has periodicity $a(t) = a(t+2\pi L)$,

$$3L^2(S^1(R\mathbb{L})) \quad E_n = \left( \frac{n}{R\mathbb{L}} \right)^2 \frac{\omega^2}{L^2}$$

$w$ = winding of string around $S^1$,

$n$ = momentum of string of mass around $S^1$.

Exchange $R \rightarrow \frac{1}{R}$ is like exchanging $n \rightarrow \omega \rightarrow n \ 	ext{-- "T-duality"}$

```
Remark: The action $S(\phi) = \int \frac{1}{2} \not{d}x \not{d}t \phi \not{d}^4 \phi = \frac{1}{2} \int \phi \not{d}^4 \phi$ is conformally inv. \ 
All energy of states are $\omega \frac{1}{L}$
```

$$Z = \sum_n e^{-\omega^2 n^2 / L^2} \quad E_n = \frac{\omega^2}{L^2}$$

$$Z = \sum_n e^{-\omega^2 n^2 / L^2} \quad \text{scale inv. (inv. under } T \rightarrow 2T, L \rightarrow 2L)$$
The bigger theory:

\[ S_{big} = \int \frac{1}{2} \sum_{i=1}^{n} |A_i|^2 + \frac{1}{4\pi^2} \int \mathbf{B} \cdot d\mathbf{q}, \quad Z = \int \mathcal{D}\mathbf{p} \mathcal{D}\mathcal{B} e^{-S_{big}} \]

Option 1: eliminate \( \mathcal{B} \)

\[ S_{big} = \frac{1}{2\pi} \int \int \frac{1}{2} \left( \mathbf{B} - 2 \mathbf{R} \right) \cdot d\mathbf{q} + \frac{1}{4\pi^2} \int \mathbf{B} \cdot d\mathbf{q} \]

Let \( \mathbf{R} = \mathbf{B} - 2 \mathbf{R} \) and solve \( \mathbf{B} \) is Gaussian.

- \( \mathbf{R} \) contributes some \( m \)-dim. det. indep. of \( \mathbf{p} \), what remains is

\[ S_{eff} = \frac{R^2}{4\pi} \int \mathcal{D}\mathbf{q} \mathcal{D}\mathbf{p} e^{-\frac{1}{4\pi^2} \int \mathbf{B} \cdot d\mathbf{q} \cdot \mathbf{B}} \] - periodicity \( \mathbf{R} + \mathbf{Z} \)

Option 2: \( \mathbf{R} \) over \( \mathbf{p} \), \( \mathbf{q} \) only appears linearly in \( S_{big} \),

Recall:

\[ \int \mathcal{D}\mathcal{B} e^{\frac{1}{2\pi} \int \mathbf{B} \cdot d\mathbf{q}} = \mathcal{N}, \quad \sum \mathcal{D}\mathbf{p} \mathcal{D}\mathbf{q} \mathcal{N} = \mathcal{N}(\mathbf{p}, \mathbf{q}) \]

\[ \int \mathcal{D}\mathbf{p} e^{\frac{1}{2\pi} \int \mathbf{B} \cdot d\mathbf{q}} = \int \mathcal{D}\mathbf{p} e^{\frac{1}{2\pi} \int \mathbf{B} \cdot d\mathbf{q}} \]

Expanded:

\[ \mathcal{D}\mathbf{p} = \mathcal{D}\mathbf{p}_0 + \sum_{i=1}^{n} \mathcal{D}\mathbf{p}_i, \quad \mathcal{D}\mathbf{q} = \mathcal{D}\mathbf{q}_0 + \sum_{i=1}^{n} \mathcal{D}\mathbf{q}_i \]

Integrating over \( \mathbf{p} \) means \( \forall \mathbf{p}_0, \exists \mathbf{p}_0 \) such that \( \mathbf{p}_0 = \mathbf{p}_0 \).

Integration over \( \mathbf{q}_0 \):

\[ \frac{1}{2\pi} \int \mathbf{B} \cdot d\mathbf{p}_0 = \frac{1}{2\pi} \int \mathbf{B} \cdot d\mathbf{p}_0 \]

- \( \mathbf{q}_0 \) produces \( \delta \)-func.

\[ \int \mathcal{D}\mathbf{q}_0 e^{\frac{1}{2\pi} \int \mathbf{B} \cdot d\mathbf{q}_0} = \delta(\mathbf{B}) \]

- Gives \( \mathbf{R}_0 + \mathbf{Z} \).

- Write \( \mathbf{R}_0 = \mathbf{R}_0 + \mathbf{Z} \).

Play this in effective action:

\[ S_{eff} = \frac{1}{4\pi} \int \mathbf{B} \cdot d\mathbf{q} + \frac{1}{2\pi} \int \left( \sum_{i=1}^{n} \mathbf{R}_{0i} \cdot d\mathbf{q}_i + \sum_{i=1}^{n} \mathbf{B} \cdot d\mathbf{q}_i \right) \]

- ...
\[ \sum_{\nu_i} \sum_{k \in \mathbb{Z}} e^{2\pi i \nu_i \cdot k} = \sum_{k \in \mathbb{Z}} d(a^i - k) \]

T-duality:

\[ S'(k) = \frac{1}{4\pi^2} \int \| \eta \|^2 = \frac{1}{4\pi} \int \| \frac{d}{dR} \eta \|^2. \]

T-duality:

1) simplest prototype for mirror symmetry (based on a similar equivalence between 2 diff. z-models with targets X, Y)
2) analogous phenomena in 4-dim gauge theory, electric/magnetic duality.

\[ \text{Pachian gauge theory in 4-dim.} \]

\[ X \text{ is oriented 4-manifold,} \]

\[ E = \left\{ \begin{array}{c}
\text{principal U(1)-bundle over X}
\text{U(1)-connection in } P
\end{array} \right\} \]

\[ F = \text{curvature of } \nabla, \quad F \in \mathfrak{so}^*(\mathfrak{g}). \]

Fix \( \delta S' = 0 \).

\[ \delta S = \frac{i}{2g^2} \int_X \mathcal{F} \wedge F + \frac{i \delta g^2}{4\pi^2} \int_X \mathcal{T} \wedge \mathcal{F} \]

\[ \delta \text{ vanishes if } \delta \mathcal{F} = 0, \]

\[ \text{So, classical eq. of motion: } d \mathcal{F} = 0 \]

\[ \text{Riem. } \int \frac{d}{dR} = 0 \]

\( \int \frac{d}{dR} \) \text{ are source-free Maxwell's equations as learned in HS (high school)}

\[ \mathcal{E} = \nabla \mathcal{F}, \quad \mathcal{F} = \frac{i}{e} (\mathcal{F} \wedge \mathcal{F}) \]

\[ \delta S = \frac{i e}{4\pi} \int_X \mathcal{F} \wedge \mathcal{F} - \frac{i e}{4\pi} \int_X \mathcal{F} \wedge \mathcal{F} \]

\[ \frac{4\pi}{\mathcal{E}} \]

\[ \mathcal{R} \quad \text{d} \mathcal{F} \]

\[ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \]

\[ \frac{1}{i e} \quad \mathcal{R} \quad \mathcal{F} \]

\[ \text{also: } e = e^{-z} \]

\[ \begin{array}{c}
\frac{4\pi}{\mathcal{E}}
\end{array} \]

\[ \langle \mathcal{E}_k(x_1, x_2, \ldots, x_n) \rangle = \int \mathcal{D} \zeta \zeta \cdot e^{-S} \]

\[ \langle \mathcal{E}_k \rangle = \frac{1}{z} \]

\[ \langle \mathcal{F} \rangle = \frac{1}{z} \]
Why \(U(1)\)-connections instead of just 1-forms?  
\[ F = \begin{pmatrix} 0 & i \varphi \tau^1 \\ -i \varphi & 0 \end{pmatrix} \]  
\[ \begin{align*} \text{div} \, F &= 0 \\ \text{curl} \, F &= 0 \end{align*} \]

**Hamonic Tens.**  

\[ \begin{align*} \text{Hodge} & \quad (\text{Helmholtz's notation}) \\ \text{curl} \, E &= -\frac{\partial B}{\partial t} \\ \text{curl} \, B &= \frac{\partial E}{\partial t} \end{align*} \]

\[ S = \frac{1}{4} \int_X \frac{\sqrt{\text{det} } \, F}{4\pi^2} \int_X \sqrt{\text{det} } \, F \\ = \frac{i \varphi}{4\pi} \int_X \sqrt{\text{det} } \, F \, \varphi \, - \frac{i \varphi}{4\pi} \int_X \sqrt{\text{det} } \, F \, \varphi \]

**Remark:**  
1. The theory \( W \) coupling \( \varphi \) is equivalent to the theory of coupling \( \varphi + \theta \), i.e., \( \theta \rightarrow \theta + \varphi \)  
   because \( \varphi \) is invariant under this shift.

2. Also have an equivalence \( \varphi \leftrightarrow -\frac{1}{\varphi} \) (this has path \( 0 \to \varphi \) like \( \text{Re} \to \frac{1}{\text{Re}} \))

3. Appearance of \( \varphi \in \mathbf{H} \) on upper \( \mathbb{C} \)-plane  
   \[ \mathbb{C} \to \{ (\mathbb{R}, \mathbb{R}+\mathbb{C}) | \mathbb{R} \leq 1/2 \} \]

\[ \text{Is there a torus around?} \]

**Claim:** Yes, there exists a 6-dim \( M_{\text{BSY}} \) on \( X \) with coupling \( \varphi \)
Classically, an electric charge/current
\[ j = \left( \begin{array}{c} j_1 \\ j_2 \\ j_3 \end{array} \right) \in \mathbb{R}^3 \]

\( d^* j = 0 \) modifies Maxwell eq. to
\[ \begin{align*}
\text{d} t &= 0 \\
\text{d} A &\equiv \text{d} A^+ = 2 j^+ \\
\end{align*} \]

To write a \( \text{d}^* \text{d}T \) of \( \mathbb{R}^3 \) fields interacting w/ this fixed \( j \), have to add a term to \( S \) locally taking the form:
\[ S_{\text{gauge}} = \int A \wedge \theta \]
\( \theta \in \Omega^2(\mathbb{R}) \) because \( \text{d} \theta = 0 \), \( \theta = d \phi \).

If \( \Phi \) is trivial, can literally do this.

Ex.: Show in this case \( S_{\text{gauge}} \) is indeed the change of the choice of trivial \( \Phi \). (was \( d^* j = 0 \))

**Gauge invariance**

\[ \mathcal{C} = \{(P, \Phi) : P \text{ a } GL(n, \mathbb{R}) \} \]

To compute path-$\mathcal{C}$: Attempt to make \( \mathcal{C} \) concrete: Fix on \( GL(n, \mathbb{R}) \)-bundle \( P \), in each equiv. class \( i \), then solve all connections \( D \) on \( P \). Still have to identify \( D \) differing by \( \text{Aut}(\text{equiv. classes of } P) \)

Concretely, \( D \mapsto P + \text{d} \phi \), \( \phi : x \mapsto \phi(x) \)

\( \phi \) is inv under this.

After adding \( S_{\text{gauge}} \), is gauge invariance retained?
\[ \int A \wedge \theta = \int (A+\text{d} \phi) \wedge \theta = \int A \wedge \theta \]

if \( \phi \) lifts to \( \bar{\phi} : x \mapsto \bar{x} \).

For general \( \bar{x} : x \mapsto \bar{x} \) we \( \bar{\phi} \) have gauge invariance iff \( \Delta \bar{x} \in \mathbb{Z} \bar{x} \) ("quantization of charge")

An example: Pick \( \text{an} \) \( \bar{\phi} : x \mapsto \bar{x} = x + \delta x \), dual current \( \delta x \) - distribution valued 3-form.

\[ \int_C \delta \phi = \nu \wedge C \]

\( \delta \phi \) draw
So a good example of $j$ is

$$A_j = \sum_{i} \delta_{ij} \eta_i, \quad \eta_i \in \mathbb{Z}.$$  

Gauss's law:

$$\oint\int B \cdot d\mathbf{S} = \int \int_{\mathcal{B}} \mathbf{E} \cdot d\mathbf{S}.$$  

$$\sum_{\mathcal{B}} F = \sum_{\mathcal{B}} \mathbf{S}.$$  

Remark: If $R$ is not trivial, not as easy to write the coupling to $j$. See notes for how to do it using path identity, if $\phi j$ is exact.

If $\eta_j = \delta_{ik}$, $k \neq 0$, money we can think of adding $\sum_{k=1}^{n+1} \eta_k \phi_k$, as multiplying $e^{-i} \sum_{k=1}^{n+1} \eta_k \phi_k$ by $e^{-i} \sum_{k=1}^{n+1} \eta_k \phi_k$.

i.e., invert an observable $\phi \gamma_i (v)$. When line $v$ in eq. $v_i \to \text{trajectories of charged particle}$.

$$\nu = \frac{\mu \times \text{mag.}}{\alpha}.$$  

**Coupling to dynamical fields**

$$C = \| \mathcal{P} \| \text{W-bundle}$$

$$\mathcal{P} \text{can. in } \mathbb{K}$$

$$\mathcal{P} \text{section of associated bundle } E_\mathbb{K} = \mathcal{P} \times \text{U}(1)$$

$\mathcal{P}$ induces a connection $\mathcal{D}$ in $E_\mathbb{K}$. In loc thv. of $\mathcal{P}$, $\mathcal{D}$ is trp. by $\mathcal{D} \mathcal{P} = \mathcal{P} \mathcal{D} \mathcal{P}$.

$$\mathcal{D} \mathcal{P} = (d\mathcal{P} + i k \mathcal{P}) \in \mathcal{D} \mathcal{P} \times \text{U}(1).$$  

$$S = \sum_{\gamma} \left( V - \frac{1}{2} \int \mathcal{D} \mathcal{P} \mathcal{D} \mathcal{P}^* - \frac{1}{2} \int \mathcal{A}^* \mathcal{A} - \frac{1}{2} \int \mathcal{A}^* \mathcal{A} \right)$$

$$+ \frac{2}{n} \int \int \mathcal{E} \phi \mathcal{E} \phi^* - \frac{2}{n} \int \int \mathcal{E} \phi \mathcal{E} \phi^*$$

$$\text{Gauge inv. of motion now say } \frac{d}{dt} \mathcal{F} = \frac{\partial}{\partial \mathcal{F}} \left( \int \int \mathcal{D} \mathcal{P} \mathcal{D} \mathcal{P}^* \right), \quad \mathcal{D} \mathcal{P} = \mathcal{D} \mathcal{P}.$$  

$d\mathcal{P} = 0$ due to Bianchi identities.

"Said QED". EM field coupled to massive, electrically charged field.

**Remark:** Nobody knows how to write an action that includes also magnetically charged particles as fields.

For perturbation theory (expansion around $g = 0$) first note that

$$O \leq \frac{1}{2} \int \mathcal{E} \phi^* \phi \mathcal{F} + \frac{1}{2} \int \mathcal{E} \phi^* \phi \mathcal{F} + \mathcal{F} \phi \phi^*,$$

$$\frac{1}{2} \int \mathcal{E} \phi^* \phi \mathcal{F} + \frac{1}{2} \int \mathcal{E} \phi^* \phi \mathcal{F},$$

$$\rightarrow \int \mathcal{F} \phi \phi^*.$$  

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So the path-fermion sector with \( \int d^4\phi \sim e^{\frac{g^2}{2}} \) has \( S \approx \frac{g^2}{2} k^2 \) 

\[ e^{-S} \sim e^{-2g^2 k^2} \] 

an expansion around \( g = 0 \), 

\( \Rightarrow \) contributions from sectors with \( a \neq 0 \) invisible in series expansion around \( g = 0 \).

(Slogan: "Infrared is a non-perturbative effect").

Q: Why does \( g \) behave like a coupling?

\[
S = \frac{1}{2} \int F^2 + \ldots + 3g \bar{\psi} \gamma_5 \psi + \ldots \\
\psi \rightarrow g \phi \\
\Rightarrow S = \frac{1}{2} \int (\phi^2) + \ldots + g \bar{\phi} \gamma_5 \phi 
\]

Last time: \( U(1) \) gauge theory + charged scalar

dim \( k = 4 \) 

\( \phi \) in non-triv. rep. of \( U(1) \)

\[
S(\phi) = \frac{1}{g^2} \int \left( \frac{1}{2} \partial \phi \cdot \partial \phi + \frac{\lambda}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 \right) \\
\phi \rightarrow e^{ik \cdot \phi} \\
\Rightarrow \text{DE: } \delta \phi = \delta \phi + k A \delta \phi
\]

Because \( S \) isn't just quadratic, anything we compute will involve interactions - not just \( S \).

Let's look at a simple toy model of a 4-d theory w/ interactions.

\( \phi \in \mathbb{K}^4 = \text{Map}(X, \mathbb{R}) \)

\[
S(\phi) = \int \frac{1}{8} \partial \phi \cdot \partial \phi + \frac{\mu^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 \\
\phi \rightarrow e^{i \phi} \\
\Rightarrow \text{DE: } \delta \phi = \delta \phi + k A \delta \phi
\]

Q: If dim \( k = 1 \), this is the harmonic oscillator.

\( X = \mathbb{R}^4 \). Let's try to compute \( \langle \phi(\xi) \phi(0) \rangle \) up to order \( \lambda_1 \)

Rules of perturbation theory: up to order \( \lambda_1 \)

\[
\langle \phi(x_1, 0) \phi(0) \rangle \sim \frac{(\xi_1 \cdot \partial)^2}{2} a^{(0)} (x_1, x_2) \in \mathbb{R}^4 \text{ (Euclidian)} \\
\Rightarrow \text{DE: } \delta \phi = \delta \phi + k A \delta \phi
\]
\[ D(x,y) = \text{Green's Sea of operator } (-\Delta + \mathbb{I}) \]
\[ (-\Delta + \mathbb{I}) \delta(x-y) = \delta(x-y) \]

(Recall: For \( x\in\mathbb{R}^n, u=0 \) have \( \delta(x-y) = \frac{1}{|x-y|} \))

For \( x\in\mathbb{R}^n, u=0 \)
\[ \delta(x-y) = \frac{1}{|x-y|^{n-1}} \quad (\text{up to const.}) \quad (\text{max gen.: } \frac{1}{|x-y|^{n-1}}) \]

How to interpret \( \delta \)?

Suppose we tried defining this 4-th theory by latitication.

One important effect of this: "UV cut-off" - fields of definite on the lattice have \( \phi(p) \) cut-off, \( \phi(p) = e^{ip \phi(x)} \)

\[ \phi(p) = 0 \text{ for } |p| > A, \quad A = \frac{4}{\lambda} \]

\[ \mathcal{L} = \frac{1}{2} \sum_{\phi, \phi'} \left[ \phi(x), \phi(x') \right] \]

Cut-off Theory: For \( \Lambda \in \mathbb{R}^n \)

\[ \mathcal{L} = \left\{ \phi: \mathbb{R}^n \to \mathbb{R} / \phi(p) = 0 \quad \forall |p| > \Lambda \right\} \]

In the cut-off theory, \( D(x,y) \) will be replaced by \( D_\Lambda(x,y) \)

\[ D_\Lambda(x,y) = \int \frac{dp}{(2\pi)^n} e^{ip(x-y)} \]

\[ D_\Lambda(x,x) = \int \frac{dp}{(2\pi)^n} \frac{1}{|p|^2 + \Lambda^2} \propto \int \frac{dr}{r^{n+2}} \text{ diverges.} \]

So: the divergence of \( D(x,y) \) comes from the region of large \( p \). With the cut-off,

\[ D_\Lambda(x,y) = \int \frac{dp}{2\pi^2} \frac{e^{ip(x-y)}}{|p|^2 + \Lambda^2} \quad \text{and} \quad D_\Lambda(x,y) \text{ finite.} \]

\[ \delta(x-y) = \frac{1}{|x-y|^{n-1}} \quad (\text{max gen.: } \frac{1}{|x-y|^{n-1}}) \]

\[ \mathcal{L} = \frac{1}{4\pi^2} \frac{1}{|x-y|^{n-1}} \quad \text{for } \lambda \to 0 \]

Further, this term can be \( \delta \text{-term} \) even for \( \Lambda \text{ small} \).
For $\lambda \gg e^{-\frac{1}{\lambda^2 r}}$, this expansion is ill-behaved.

What do we do about this problem?

Try to formulate computations using an effective action $S_{\text{eff}}(\lambda')$ that the cut-off is $\lambda'\approx E$.

To get $S_{\text{eff}}(\lambda')$ from our original $S$, need to integrate out all the modes $\phi(p)$ for $\Lambda<|p|<\Lambda'$.

General expectation: $S_{\text{eff}}(\lambda')$ is non-local; expanding it in powers of fields and derivatives, get infinitely many terms.

$\rightarrow$ Looks dangerous: How to use it?

Convenient way to investigate: define an $n$-dim. space of all possible actions $S(\phi)$ (vary under $\phi \rightarrow \phi'$)

Now define a flow on $\phi$:

$\tau \rightarrow \frac{1}{\tau} \int \frac{d^4x}{(2\pi)^4} \sqrt{-g} e^{-\frac{\tau}{2} \int \text{Ric}}$

$\tau \rightarrow \frac{1}{\tau}$ = effective actions obtained by $\int$ out modes $\Lambda e^{-\tau} < |p| < \Lambda'$

Call

$\lambda' :$ Polchinski, "renormalization and effective Lagrangians"

Key claim: As $\tau \rightarrow 0$, this flow is driven to a 3-dim. space! (in this specific theory) $(\phi^1, \phi^2, \phi^3)$

To understand why: Scaling $\phi \rightarrow \varepsilon \phi(x)$ leaves the term $\frac{1}{2} \int \text{Ric}$ invariant.

Transforms the other terms:

$S(\varepsilon^2 (\int \text{d}^4 x \phi^4)) = \varepsilon^{-4} \int \phi^4$

$S(\int \text{Ric}) = \varepsilon^{-2} \int \text{Ric}$

Define scaling dimension of the coupling: $\dim(\phi^2) = n$

Formally (we neglect):

$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \langle \phi(x) \phi(y) \rangle \rightarrow \text{compared w/ action } S$

$\varepsilon_{\Lambda'}^n(\phi^2) \rightarrow \lim \int d^{4n} \phi^n$

$= \rightarrow$ formally, for large $\varepsilon$ (large distance), the effect of the terms w/ $\dim > 4$ is small. Call these terms "irrelevant."
Last time: scalar field theory in $d=4$, cutoff $\Lambda$ (all four fields with $\Phi^2 = 0$ for $\Phi > \Lambda$)

\[ S(\Phi) = \int d^4x \left( \frac{1}{2} \partial \Phi^2 + \frac{m^2}{2} \Phi^2 + \frac{\lambda}{4!} \Phi^4 \right). \]

Then

\[ \frac{\langle \Phi(x_1) \Phi(x_2) \rangle}{2} \propto \int \frac{d^Dx}{x_1 - x_2} \lambda^\Lambda \Lambda \rightarrow \infty \]

\[ \mathcal{D}_\Lambda^2(x_1, x_2) = \int d^Dk \frac{e^{i k \cdot (x_1 - x_2)}}{m^2 + \pi \Lambda^2} \]

Large-$\Lambda$ behavior of the terms:

\[ N \mathcal{D}_\Lambda(x_1, x_2) \propto \Lambda + \ldots \]

\[ \left( \frac{i}{\Lambda^2} + \ldots \right) \]

So "subleading term >> leading term if $\Lambda^2 \ll \Lambda^{-1}$

\[ \text{What are we headed to?} \]

One specific 4-d QFT: $N=2$ SYM YM theory in 4-dim, $U(1)_W$ gauge group $G = SU(2)$.

Fields:

- $P$ principal $SU(2)$-bundle over $x, \omega$ connection
- $\Theta$ section of $P \times G = \text{adjoint}$
- Spin bundle $\Gamma(x) \mathcal{R}$ vector space

\[ S = \frac{1}{g^2} \int \frac{1}{4} \text{tr} \frac{\partial \Theta^2}{\partial \omega^2} + \frac{1}{g^2} \text{tr} \Gamma \mathcal{F} \Gamma + \frac{1}{2} \int \text{tr} \frac{\partial \Theta^2}{\partial \omega^2} + \text{(terms involving $\Theta$)} \]

"YM action" (generalizes $U(1)$ case

Key property: when $k = 0$, $Z_k$ admits odd vector fields \( Q \) with $\langle Q \rangle = 0$. ("supersymmetry")

Refers: the Poincaré algebra $so(4)$ gets extended to a super Lie algebra.

\[ \tilde{\delta} = -i \lambda^a \delta \frac{\delta}{\delta \phi^a} \]

$\text{so}(4) = \text{so}(3) \oplus \mathbb{R}^3$.

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It's believed that for QFT defined by this action! i.e., can define it by putting it on a lattice, taking L→0 (or take cutoff, taking Λ→∞). (Not true, e.g., for $S = \frac{1}{2} \text{Tr} (\phi^2 + \phi^4)$)

(a) What's idea 1: study this theory on $X = \text{opt} \text{ Poincare 4-manifold with spin structure.}$

- If you do this, no iso(4), no $O_g$

- $Q_x$, $Q_x$

(b) What twisting means: replace the vector space $\mathbb{R}$ by a vector bundle over $X$. don't need spin structure.

- $R = S^+$. So now $\psi$ is a section of $(\mathbb{C}^2 \otimes S \otimes S^+)$

Consequences of the odd symmetry $Q$: (w $Q^2 = 0$)

$Q^2 = 0$

and $Q$-closed operators

$Z$ and correlation funs are invariant under change $S \rightarrow S + Q(\phi)$

they are formally independent of the metric on $X$!

(Computation: look at $\frac{\delta S}{\delta \phi}$ and see that it's of the form $Q(\phi)$)

$\Rightarrow$ should get topological invariants of $X$: 

$Z = \langle \chi, (Q^1, \ldots, Q_n) \rangle$ where each $Q_i$ has $Q_i Q_i = 0$.

What are the allowed $Q_i$?

E.g., local operators $Q(x)$: $Q(x) = \text{Tr} \left[ \phi \phi^* \right] \Rightarrow Q \phi = 0$

$\Rightarrow$ in representation

Also have $Q$-inv. non-local operators:

$Q_x^{\text{top}}$

$Q^{(S)}_{\text{surface}}$

$Q^{(M)}_{\text{chain}}$

$Q^{(C)}$
So, Witten says: Get a top. inv. for a V4-wld 0 inv. closed \[
\begin{array}{c}
\text{0-wlds} \\
\text{1-wlds} \\
\text{2-wlds}
\end{array}
\].

Claim: There are \(=\) to invariants induced by Donaldson.

Why? Compute by localization \(\rightarrow\) make a deformation \(S \to S + t Q^P\), take limit \(t \to 0\):

\[
\text{get localized to a f.d. integ. integral over induced model space} \quad (\text{e.g. of } SU(2) - \text{ann. Dwy}, F_0^+ = 0)
\]

Top. info of the data, defining the QFT, e.g.,

1) 0-d data \(f(a)\)
   \[
   \text{inv.} = \sum_{a} \frac{1}{f(a) - 0}
   \]

2) 1-d data \(\text{Riem. spin wld } y\)
   \[
   \text{inv.} = \text{ind} (\Theta; S_y - S_y)
   \]

3) data \(\text{symplectic wld } y\)

\[
\text{Hamiltonian inv. action}
\]

\[
\text{inv.} = \int \text{vol} \, e^{-h} \quad \text{computed by localization}
\]

Now, for theory: data \(=\) Lie group \(G = SU(2) + \text{space time } x \in \text{man. 4-wld}\)

We'll get top. inv. of \(X\).

Effective field theory: QFT

\[
\int \text{out some of the fields } (C)
\]

\text{simper QFT}

Main themes: 1) \(\text{SUSY} \) often compute topological inv. of their data \(\text{localization}\) \(\Rightarrow\) \(\text{xxx}\)

2) QFTs can often be replaced by simpler ones, which computes the same quantities in low energy approx., i.e., compute some correlation \(\text{xxx}\)
\[ \langle \phi(t_1), \phi(t_2), \ldots, \phi(t_n) \rangle \]

\[ \text{as } t \to 0 \]

\[ \langle \phi(t_1), \phi(t_2), \ldots, \phi(t_n) \rangle \to \rho \phi(t_1) \cdots \phi(t_n) \]

\[ \text{Why this helps us:} \]

We study 4d twisted \( N=2 \) SYM on opt Picard 4-manifold \( X \). \( \mathbb{Z}_2 \langle 0 \rangle \) are top. inv. of \( X \).

In particular, inv. under \( g \to -g \).

In large \( t \) limit, should be able to compute \( \mathbb{Z}_2 \langle 0 \rangle \) exactly using the low-energy limit of \( N=2 \) SYM.

\[ \mathcal{C} = \left\{ \begin{array}{l}
\phi \in \Gamma(\text{ad} P_{x_0}) \\
\lambda^\pm \in \mathbb{D} \Gamma / S^2 \otimes (\text{ad} P_{x_0} \otimes \mathbb{R}) \\
\sigma \in \Gamma(\text{ad} P_0 \otimes \text{Sym}^2 \mathbb{R})
\end{array} \right. \]

Expand w.r.t. basis of \( R \): \( \lambda^\pm \to \lambda_1^\pm, \lambda_2^\pm \)

\[ S = \frac{1}{g^2} \int_X \left( -\frac{1}{4} F \wedge F + \nabla_\phi \nabla_\phi - i \delta^\nu \nabla_\nu \phi \phi - i \delta^\nu \nabla_\nu \phi \phi + \frac{i}{4} \delta^\nu \nabla_\nu \phi \phi + i \delta^\nu \nabla_\nu \phi \phi + \frac{1}{g^2} \delta_{\lambda, \lambda} \nabla_{\lambda} \nabla_{\lambda} - \frac{\beta}{2} \phi \phi \right) \]

\[ \mathcal{S} = \frac{1}{g^4} \int_X \left( \frac{i}{4} \nabla^\nu \nabla_\nu \phi \phi + i \delta^\nu \nabla_\nu \phi \phi \right) + \frac{i}{4} \delta^\nu \nabla_\nu \phi \phi + \frac{i}{4} \delta^\nu \nabla_\nu \phi \phi + \frac{i}{4} \delta^\nu \nabla_\nu \phi \phi + \frac{i}{4} \delta^\nu \nabla_\nu \phi \phi \]

\[ \zeta \mid = - \zeta \mid, \quad \zeta \mid = 0 \]

Odd symmetries: \( \tilde{\lambda}^\pm \equiv \lambda^\pm \)

Odd vector fields \( Q_\xi, \xi \in (S^+ \oplus S^-) \otimes \mathbb{R} \).

\[ \delta \phi = \int \tilde{\zeta} \left( \nabla^\nu \phi \phi \right) \]

\[ \delta A_\mu = \delta (i \left( \nabla_\mu \phi \phi \right) - \nabla_\mu (\phi \phi)) \]

\[ \phi^\mu = \text{Clifford action } \mu \otimes S^\pm \to S^\pm \]
If \( x \neq \mathbb{R}^4 \), could try taking \( \xi \), a section of \((S^+_x \otimes S^-_x) \otimes R\) and write the same formula for \( Q_x \) on \( \mathbb{R}^4 \). But then, \( Q_x S = 0 \iff \xi \) is covariantly constant section of \((S^+ \otimes S^-) \otimes R\).

For a 4-manifolm \( X \), a covariantly constant spinors \( \xi \), \( \nabla_x \xi = 0 \iff \xi = 0 \).

Q: When does \( X \) admit a covariantly constant spinor \( \xi \)? \( \xi = 0 \iff Q_x (e^-) = 0 \) for action is constant.

Twisting. Replace \( R \) by \( \mathbb{S}^5 \) everywhere. In particular, now odd vector bundles \( Q_x \) labeled by \( \xi \), section of \((S^+ \otimes S^-) \otimes (\mathbb{S}^5 \otimes S^+)) \).

\[
2x2 = 1 + 3 = \big( \text{triv.} \otimes \text{ Sym}_2 (S^+) \big) \oplus V
\]

\[s(2), \ \sigma(2) \times s(2) \quad \text{Rep. of Spin}(4), \text{ descend to Spin}(4). \]

Now odd symm. \( Q_x S = 0 \iff \xi \) cov. const.

Taking \( \xi \) to be a cov. const. section of the trivial summand in \((S^+ \otimes S^-) \otimes S^+ \) (i.e., const. form), get odd symm. of \( S \), \( Q S = 0 \).

The twisted theory:

\[
\mathcal{Z} = \left\{ \begin{array}{ll}
\phi \in \Gamma \left( \text{ad}_\rho \otimes \Lambda^+ \right) & \text{even} \\
\eta \in \Gamma \left( \text{ad}_\rho \otimes \Lambda^- \right) & \text{odd}
\end{array} \right.
\]

\[
S = \frac{1}{8} \left[ \frac{1}{i} \text{Tr} \left( i \Phi \mathcal{P} - i \left( \Phi, \mathcal{P} \right) - \frac{i}{4} \mathcal{P} \Phi \mathcal{P} \Phi - \frac{1}{2} \| \mathcal{P} \|^2 + \frac{1}{4} \| \Phi \|^2 - \frac{1}{4} \left[ \phi, \phi^* \right] \phi^* - \frac{1}{4} \left( \Phi, \phi^* \phi \right) \right) + \frac{2}{3} \gamma \int \text{Tr} (\Phi, \mathcal{P} \Phi) \right]
\]

\[
\mathcal{A} = \frac{1}{2} \int \left( \text{Tr} \left( \Phi \mathcal{P} - i \left( \Phi, \mathcal{P} \right) - \frac{i}{4} \mathcal{P} \Phi \mathcal{P} \Phi - \frac{1}{2} \| \mathcal{P} \|^2 + \frac{1}{4} \| \Phi \|^2 - \frac{1}{4} \left[ \phi, \phi^* \right] \phi^* - \frac{1}{4} \left( \Phi, \phi^* \phi \right) \right) + \frac{2}{3} \gamma \int \text{Tr} (\Phi, \mathcal{P} \Phi) \right]
\]

\[\xi \in \mathcal{P}_+ \otimes S^+ \rightarrow S \otimes S^+ \]

\[\xi \in \mathcal{P}_- \otimes S^- \rightarrow S \otimes S^- \]
\( Q \) acts by
\[
\delta \phi = 0, \quad \delta \overline{\phi} = \epsilon A \overline{\phi}, \quad \delta A = \epsilon \phi, \quad \delta \eta = \epsilon [\phi, \overline{\phi}], \quad \delta \phi_\mu = \epsilon \partial_\mu \overline{\phi},
\]
\[
\delta \chi = \epsilon (\mathcal{F}^- - D), \quad D = \epsilon (2\psi) + 2\epsilon [\phi, \overline{\phi}].
\]

Before, \( S \rightarrow S + \epsilon \psi \), and as \( \epsilon \rightarrow 0 \), get localization to order of terms \( \mathcal{O}(\epsilon) \).

Now look at vanishing locus of \( Q \). This requires that the above are zero; more precisely
\[
\Phi \overline{\Phi} = 0, \quad \mathcal{F}^+ = 0, \quad \mathcal{F}^- = 0
\]

\[\text{generically}\]
\[\Phi \overline{\Phi} = 0, \quad \mathcal{F}^+ = 0\]

---

**SU(2) \rightarrow U(1) (\frac{\omega}{2\pi})**

Will induct, only occurs if \( 3 \mathfrak{g}-\text{form} \), \( \omega \) anti-self-dual and \( \omega \in \mathcal{H}(S^2, \mathbb{Z}) \)

---

A \( b_2 \rightarrow 1 \) and metric on \( X \) generic, then no such \( \phi \)