

Galois sections for abelian varieties over number fields

MIRELA ÇIPERIANI AND JAKOB STIX

Abstract — For an abelian variety A over a number field k we discuss the space of sections of its fundamental group extension $\pi_1(A/k)$. By analyzing the maximal divisible subgroup of the Weil–Châtelet group $H^1(k, A)$ we show that the space of sections of $\pi_1(A/k)$ contains a copy of $\hat{\mathbb{Z}}^{[k:\mathbb{Q}] \cdot \dim(A)}$ and is never in bijection with $A(k)$. This is essentially a result about the structure of $H^1(k, T_\ell(A))$.

1. GALOIS SECTIONS FOR ABELIAN VARIETIES

1.1. **Galois sections.** The étale fundamental group $\pi_1(X, \bar{x})$ of a geometrically connected variety $\text{pr} : X \rightarrow \text{Spec}(k)$ forms an extension, see [SGA1] IX Theorem 6.1,

$$1 \rightarrow \pi_1(X_{\bar{k}}, \bar{x}) \rightarrow \pi_1(X, \bar{x}) \xrightarrow{\text{pr}_*} \text{Gal}_k \rightarrow 1, \quad (1.1)$$

where $\text{Gal}_k = \pi_1(\text{Spec}(k), \text{Spec}(\bar{k}))$ is the absolute Galois group of k with respect to an algebraic closure \bar{k}/k compatible with the geometric base point $\bar{x} \in X_{\bar{k}}$. By functoriality of π_1 for pointed spaces, the rational point $a \in X(k)$ gives rise to a continuous homomorphism

$$s_a : \text{Gal}_k \rightarrow \pi_1(X, \bar{a}) \xrightarrow{\gamma(-)\gamma^{-1}} \pi_1(X, \bar{x}),$$

where \bar{a} is a geometric point of $X_{\bar{k}}$ above a , and γ is an étale path γ from \bar{a} to \bar{x} on $X_{\bar{k}}$. We denote the set of $\pi_1(X_{\bar{k}}, \bar{x})$ -conjugacy classes of sections of (1.1) by

$$\mathcal{S}_{\pi_1(X/k)} = \{s : \text{Gal}_k \rightarrow \pi_1(X, \bar{x}) ; \text{pr}_* \circ s = \text{id}_{\text{Gal}_k}\} / \pi_1(X_{\bar{k}}, \bar{x})\text{-conjugacy}.$$

The well defined map $a \mapsto [s_a]$ is called the **profinite Kummer map**. By considering only conjugacy classes of maps we may from now on safely ignore base points. The section conjecture of anabelian geometry [Gr83] predicts, in the case of proper smooth curves over number fields of genus at least 2, that the profinite Kummer map is a bijection. We refer to [Sx13] for details of the above construction and the section conjecture in general. The main result of this note states that the profinite Kummer map is never surjective for abelian varieties.

Theorem 1. *For every abelian variety A/k over a number field k there exist sections of*

$$1 \rightarrow \pi_1(A_{\bar{k}}) \rightarrow \pi_1(A) \rightarrow \text{Gal}_k \rightarrow 1$$

that do not come from a rational point of A . More precisely, the space of sections $\mathcal{S}_{\pi_1(A/k)}$ has a group structure and it is isomorphic to $H^1(k, T(A))$ which contains a copy of $\hat{\mathbb{Z}}^{[k:\mathbb{Q}] \dim(A)}$.

Remark 2. Some remarks are in order to put Theorem 1 in perspective.

- (1) As we recall below in Section §1.2, the profinite Kummer map for an abelian variety $A(k) \rightarrow \mathcal{S}_{\pi_1(A/k)}$ factors as a map to a profinite group through the profinite completion $A(k) \otimes \hat{\mathbb{Z}}$. Therefore the first part of Theorem 1 follows by comparing the cardinality of both sides in case $A(k)$ is an infinite group. So the key difficulty is to handle abelian varieties of algebraic rank 0. The second statement on the Galois cohomology of the Tate module of an abelian variety is of independent interest.

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Let us emphasize that, for example, considering elliptic curves E/\mathbb{Q} over the rational numbers \mathbb{Q} , the case of algebraic rank 0 occurs with positive probability by [BS10] Theorem 4, and conjecturally in 50% of all cases. So the case of finite Mordell–Weil group is not an exceptional one.

- (2) Knowing the prospects of the profinite Kummer map being a bijection for abelian varieties has some significance for attempts to prove the section conjecture.

Although the section conjecture avoids a prediction for abelian varieties, the abelianization of sections by composition with the Albanese map $X \rightarrow \text{Alb}_X$ may help to understand the case of arbitrary smooth projective curves X , see [Sx13] §3.1.

In fact, the known cases of the birational section conjecture for curves over number fields use in a crucial way (after passing to a neighbourhood of the section) a map to an abelian variety of algebraic rank 0 and argue that the section there corresponds to a rational point.

- (3) The negative outcome concerning the section conjecture for abelian varieties may turn our attention also in the genuine section conjecture towards spaces of sections with local conditions, see for example [Sx12]. If we impose a Selmer condition, i.e., we require that the localisation at all places comes from a rational point, and if $\text{III}(A/k)$ has no nontrivial divisible subgroup, then the natural map

$$A(k) \otimes \hat{\mathbb{Z}} \rightarrow \mathcal{S}_{\pi_1(A/k)}^{\text{Selmer}}$$

to the space of conjugacy classes of such Selmer sections is in fact bijective. In this regard, Theorem 1 may be seen as a statement to shape our thinking about the section conjecture.

- (4) Last but not least, if we consider a finite field \mathbb{F}_q and an abelian variety A/\mathbb{F}_q , then results of Tate on Galois cohomology of such abelian varieties show that in fact $A(\mathbb{F}_q)$ is in natural bijection with the space of sections $\mathcal{S}_{\pi_1(A/\mathbb{F}_q)}$. In some sense, surprisingly, the section conjecture holds for abelian varieties over finite fields! This case also shows that unexpected things can happen in the section conjecture.

1.2. The Kummer map for abelian varieties. Let k be a field of characteristic 0. For an abelian variety A/k with Tate module

$$\Gamma(A) = \text{Hom}(\mathbb{Q}/\mathbb{Z}, A(\bar{k})) = \varprojlim_n A_n(\bar{k}),$$

where A_n denotes the n -torsion of A , we have a natural identification

$$\mathcal{S}_{\pi_1(A/k)} = \text{H}^1(k, \Gamma(A)) = \varprojlim_n \text{H}^1(k, A_n)$$

under which the profinite Kummer map $\kappa : A(k) \rightarrow \mathcal{S}_{\pi_1(A/k)}$ factors over the limit of the maps

$$\delta_n : A(k) \rightarrow \text{H}^1(k, A_n)$$

of the multiplication by n sequence for A , see [Sx13] Corollary 71. The long exact sequence of Galois cohomology thus leads to a diagram with an exact row

$$\begin{array}{ccccccc} A(k) & \xrightarrow{\kappa} & \mathcal{S}_{\pi_1(A/k)} & & & & \\ \downarrow & & \parallel & & & & \\ 0 & \longrightarrow & \varprojlim_n A(k)/nA(k) & \xrightarrow{\varprojlim_n \delta_n} & \text{H}^1(k, \Gamma(A)) & \longrightarrow & \text{Hom}(\mathbb{Q}/\mathbb{Z}, \text{H}^1(k, A)) \longrightarrow 0. \end{array} \tag{1.2}$$

We immediately deduce the following lemma.

Lemma 3. *The profinite Kummer map for an abelian variety A/k is*

- (1) *injective if and only if $\bigcap_{n \geq 1} nA(k) = 0$, and*

(2) *surjective if and only if both the maximal divisible subgroup $\text{Div}(\mathbb{H}^1(k, A))$ of $\mathbb{H}^1(k, A)$ is trivial, and the map*

$$A(k) \rightarrow \varprojlim_n A(k)/nA(k) \quad (1.3)$$

is surjective.

Remark 4. If k/\mathbb{Q} is a finitely generated field, then the Mordell–Weil theorem and the theorem of Lang–Néron show that $A(k)$ is a finitely generated \mathbb{Z} -module. The profinite Kummer map for A/k is in fact injective over such fields by Lemma 3 (1). Moreover, if $A(k)$ is not finite, then (1.3) is not surjective and the profinite Kummer map fails to be surjective by Lemma 3 (2).

2. THE MAXIMAL DIVISIBLE SUBGROUP OF THE WEIL–CHÂTELET GROUP

2.1. Rephrasing Theorem 1. From now on let k be a number field. By Remark 4 above, the proof of the first part of Theorem 1 consists of showing that $\text{Div}(\mathbb{H}^1(k, A)) \neq 0$ if $A(k)$ is finite and thus follows from the next theorem.

Theorem 5. *Let k be a number field, and let A/k be an abelian variety.*

- (1) *If A/k has algebraic rank $r = \text{rk}_{\mathbb{Z}} A(k) < d = \dim(A) \cdot [k : \mathbb{Q}]$, then $\text{Div}(\mathbb{H}^1(k, A))$ contains a copy of $(\mathbb{Q}/\mathbb{Z})^{d-r}$.*
- (2) *The space of sections $\mathcal{S}_{\pi_1(A/k)}$ contains a copy of $\hat{\mathbb{Z}}^d$.*

2.2. Generalized Selmer groups. For a place v of k we denote the completion of k in v by k_v . We shall be working with the following generalized Selmer groups for $n \in \mathbb{N}$ or $n = \infty$. The **Selmer group free at p** is defined as

$$\mathbb{H}_{\text{Sel}_p}^1(k, A_{p^n}) = \ker \left(\mathbb{H}^1(k, A_{p^n}) \rightarrow \prod_{v \nmid p} \mathbb{H}^1(k_v, A)_{p^n} \right)$$

and the **Selmer group trivial at p** is defined as

$$\mathbb{H}_{\text{Sel}^p}^1(k, A_{p^n}) = \ker \left(\mathbb{H}^1(k, A_{p^n}) \rightarrow \prod_{v \nmid p} \mathbb{H}^1(k_v, A)_{p^n} \times \prod_{v|p} \mathbb{H}^1(k_v, A_{p^n}) \right).$$

Let A^t denote the dual abelian variety. The local conditions Sel_p for A_{p^n} and Sel^p for $A_{p^n}^t$ are dual local conditions, see [NSW08] Definition (8.7.8), and the Euler characteristic formula of Wiles, see [NSW08] Theorem (8.7.9), reads (now $n \in \mathbb{N}$)

$$\frac{|\mathbb{H}_{\text{Sel}_p}^1(k, A_{p^n})|}{|\mathbb{H}_{\text{Sel}^p}^1(k, A_{p^n}^t)|} = \frac{|A_{p^n}(k)|}{|A_{p^n}^t(k)|} \cdot \prod_v \lambda_v = \frac{|A_{p^n}(k)|}{|A_{p^n}^t(k)|} \cdot p^{nd} \cdot \prod_{v|p} |A_{p^n}^t(k_v)| \quad (2.1)$$

where the local factor λ_v is 1 for $v \nmid p\infty$, for $v | \infty$ it is $\lambda_v = p^{-n \cdot [k_v : \mathbb{R}] \dim(A)}$, and for $v | p$ by local Tate duality and Mattuck–Tate

$$\lambda_v = \frac{|\mathbb{H}^1(k_v, A_{p^n})|}{|A_{p^n}(k_v)|} = \frac{|A(k_v)/p^n A(k_v)| \cdot |\mathbb{H}^1(k_v, A)_{p^n}|}{|A_{p^n}(k_v)|} = p^{2n \cdot [k_v : \mathbb{Q}_p] \dim(A)} \cdot |A_{p^n}^t(k_v)|.$$

2.3. Proof of Theorem 5. We can focus on the p -primary part $\text{Div}(\mathbb{H}^1(k, A)_{p^\infty})$ for a prime number p . Since $\mathbb{H}^1(k_v, A)$ has finite p -part if $v \nmid p$, the group $\text{Div}(\mathbb{H}^1(k, A)_{p^\infty})$ is locally trivial outside $v | p$, in other words

$$\text{Div}(\mathbb{H}^1(k, A)_{p^\infty}) \subseteq \text{im} \left(\mathbb{H}_{\text{Sel}_p}^1(k, A_{p^\infty}) \rightarrow \mathbb{H}^1(k, A) \right). \quad (2.2)$$

The **corank** of a p -primary torsion group $M = (\mathbb{Q}_p/\mathbb{Z}_p)^m \times M_0$ with finite M_0 , is the well-defined number $m = \text{corank}_{\mathbb{Z}_p}(M)$. We set

$$\begin{aligned} s_p &= \text{corank}_{\mathbb{Z}_p}(\mathbb{H}_{\text{Sel}_p}^1(k, A_{p^\infty})), \\ s^p &= \text{corank}_{\mathbb{Z}_p}(\mathbb{H}_{\text{Sel}^p}^1(k, A_{p^\infty}^t)). \end{aligned}$$

The snake lemma applied to the defining sequences together with the finiteness of $A_{p^\infty}(k)$ and $A_{p^\infty}(k_v)$ shows that both natural maps

$$\begin{aligned} H_{\text{Sel}_p}^1(k, A_{p^n}) &\rightarrow H_{\text{Sel}_p}^1(k, A_{p^\infty})_{p^n} \\ H_{\text{Sel}_p}^1(k, A_{p^n}^t) &\rightarrow H_{\text{Sel}_p}^1(k, A_{p^\infty}^t)_{p^n} \end{aligned}$$

have kernel and cokernel bounded independently of n . Therefore there is a constant $c > 0$ independent of n such that

$$\begin{aligned} p^{ns_p}/c &\leq |H_{\text{Sel}_p}^1(k, A_{p^n})| \leq p^{ns_p} \cdot c, \quad \text{and} \\ p^{ns^p}/c &\leq |H_{\text{Sel}_p}^1(k, A_{p^n}^t)| \leq p^{ns^p} \cdot c. \end{aligned}$$

Since $|A_{p^n}(k)|/|A_{p^n}^t(k)|$ and $\prod_{v|p} |A_{p^n}^t(k_v)|$ are bounded independently of n , analyzing the asymptotic for $n \gg 0$ in (2.1) we obtain

$$s_p - s^p = d.$$

The exact sequence from Kummer theory of A/k

$$0 \rightarrow A(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow H_{\text{Sel}_p}^1(k, A_{p^\infty}) \rightarrow \text{im} (H_{\text{Sel}_p}^1(k, A_{p^\infty}) \rightarrow H^1(k, A)) \rightarrow 0 \quad (2.3)$$

splits, because $A(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ is a divisible group. Therefore (2.3) remains exact upon applying the functor $\text{Div}(-)$. By (2.2) we find a split exact sequence

$$0 \rightarrow A(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow \text{Div} (H_{\text{Sel}_p}^1(k, A_{p^\infty})) \rightarrow \text{Div} (H^1(k, A)_{p^\infty}) \rightarrow 0$$

which allows to compute coranks as

$$\text{corank}_{\mathbb{Z}_p} (\text{Div} (H^1(k, A)_{p^\infty})) = s_p - r = d + s^p - r \geq d - r$$

This shows assertion (1) of Theorem 5. For (2) we compute the rank of $\mathcal{S}_{\pi_1(A/k)}$ via (1.2) as

$$\text{rk}_{\hat{\mathbb{Z}}} \mathcal{S}_{\pi_1(A/k)} = \text{rk}_{\hat{\mathbb{Z}}} H^1(k, T(A)) = r + \text{rk}_{\hat{\mathbb{Z}}} \text{Hom} (\mathbb{Q}/\mathbb{Z}, \text{Div}(H^1(k, A))) = d + s^p \geq d.$$

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MIRELA ÇIPERIANI, DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF TEXAS AT AUSTIN, 1 UNIVERSITY STATION, C1200, AUSTIN, TEXAS 78712, USA

E-mail address: mirela@math.utexas.edu

URL: <http://www.ma.utexas.edu/users/mirela/>

JAKOB STIX, MATHEMATISCHES INSTITUT, UNIVERSITÄT HEIDELBERG, IM NEUENHEIMER FELD 288, 69120 HEIDELBERG, GERMANY

E-mail address: stix@mathi.uni-heidelberg.de

URL: <http://www.mathi.uni-heidelberg.de/~stix/>