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## Tate–Shafarevich groups in anticyclotomic $\mathbb{Z}_p$ -extensions at supersingular primes

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# Tate–Shafarevich groups in anticyclotomic $\mathbb{Z}_p$ -extensions at supersingular primes

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## ABSTRACT

Let  $E/\mathbb{Q}$  be an elliptic curve and  $p$  a prime of supersingular reduction for  $E$ . Denote by  $K_\infty$  the anticyclotomic  $\mathbb{Z}_p$ -extension of an imaginary quadratic field  $K$  which satisfies the Heegner hypothesis. Assuming that  $p$  splits in  $K/\mathbb{Q}$ , we prove that  $\text{III}(K_\infty, E)_{p^\infty}$  has trivial  $\Lambda$ -corank and, in the process, also show that  $H_{\text{Sel}}^1(K_\infty, E_{p^\infty})$  and  $E(K_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p$  both have  $\Lambda$ -corank two.

## Introduction

Let  $E$  be an elliptic curve of conductor  $N$  defined over  $\mathbb{Q}$ , and let  $p$  be a rational prime such that  $E$  has supersingular reduction at  $p$ . We denote by  $E_p$  the  $p$ -torsion of  $E$  and assume throughout the paper that  $\text{Gal}(\mathbb{Q}(E_p)/\mathbb{Q})$  is not solvable. Let  $K/\mathbb{Q}$  be any imaginary quadratic extension such that the primes dividing  $pN$  split. Denote by  $K_\infty$  the anticyclotomic  $\mathbb{Z}_p$ -extension of  $K$  which is the unique Galois extension of  $K$  such that  $\text{Gal}(K_\infty/K) \simeq \mathbb{Z}_p$  and  $\text{Gal}(K_\infty/\mathbb{Q})$  is a pro-dihedral group. We now consider the Tate–Shafarevich group of  $E/K_\infty$ , namely the group of genus-one curves defined over  $K_\infty$  with  $E$  as their Jacobian possessing a point over every completion of  $K_\infty$ ; this is a torsion group. The  $p$ -primary part of the Tate–Shafarevich group of  $E/K_\infty$ , denoted by  $\text{III}(K_\infty, E)_{p^\infty}$ , can be viewed as a module over  $\Lambda := \mathbb{Z}_p[[\text{Gal}(K_\infty/K)]]$ , and its Pontryagin dual

$$\widehat{\text{III}(K_\infty, E)_{p^\infty}} := \text{Hom}(\text{III}(K_\infty, E)_{p^\infty}, \mathbb{Q}_p/\mathbb{Z}_p)$$

is finitely generated over  $\Lambda$ . The  $\Lambda$ -corank of  $\text{III}(K_\infty, E)_{p^\infty}$  is defined to be the rank of its Pontryagin dual. We will prove the following theorem.

**THEOREM 0.1.** *The  $\Lambda$ -module  $\text{III}(K_\infty, E)_{p^\infty}$  has trivial corank.*

This result is a manifestation of the break in the behavior of the Tate–Shafarevich group at supersingular primes in comparison to ordinary primes. When  $p$  is a prime of ordinary reduction, Rubin [Rub88] (in the CM case) and Kato [Kat04] (in the non-CM case) have analyzed the behavior of the Tate–Shafarevich group over the cyclotomic  $\mathbb{Z}_p$ -extension  $\mathbb{Q}_\infty/\mathbb{Q}$ , showing that  $\text{III}(\mathbb{Q}_\infty, E)_{p^\infty}$  has trivial corank. In this same case, assuming that the primes dividing  $N$  split in  $K/\mathbb{Q}$ , Bertolini [Ber95] has shown that  $\text{III}(K_\infty, E)_{p^\infty}$  has trivial  $\Lambda$ -corank also.

When  $p$  is a prime of supersingular reduction, by using the work of Schneider [Sch85], Rohrlich [Roh84] and Kato [Kat04] one can see that the  $\Lambda$ -corank of  $\text{III}(\mathbb{Q}_\infty, E)_{p^\infty}$  is greater than or equal to one. Furthermore, under certain conditions which, in particular, imply that  $E/\mathbb{Q}$  has trivial analytic rank, Kurihara [Kur02] has proven that  $\text{III}(\mathbb{Q}_\infty, E)_{p^\infty}$  has  $\Lambda$ -corank one.

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An algebraic proof of this result has been given by Pollack [Pol05]. In this paper, we shall see that the  $\Lambda$ -corank of  $\text{III}(\mathbb{K}_\infty, E)_{p^\infty}$  is trivial, and in the process we will analyze the  $\Lambda$ -corank of the Selmer group  $H_{\text{Sel}}^1(\mathbb{K}_\infty, E_{p^\infty})$  (defined in §1) and of  $E(\mathbb{K}_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ .

### 1. Structure results

For every number field  $F$  and  $m \in \mathbb{N}$ , we can define the  $p^m$ -torsion of the Selmer group of  $E/F$  to be

$$H_{\text{Sel}}^1(F, E_{p^m}) := \ker \left[ H^1(F, E_{p^m}) \rightarrow \prod_{\lambda \subseteq F} H^1(F_\lambda, E) \right],$$

where  $\lambda$  denotes primes in  $F$  and  $F_\lambda$  is the completion of  $F$  at  $\lambda$ . Then, the  $p^m$ -torsion of the Tate–Shafarevich group of  $E/F$  fits in the exact sequence

$$0 \rightarrow E(F)/p^m E(F) \rightarrow H_{\text{Sel}}^1(F, E_{p^m}) \rightarrow \text{III}(F, E)_{p^m} \rightarrow 0. \quad (1)$$

Let  $K_n \subseteq K_\infty$  be the unique extension of  $K$  such that  $\text{Gal}(K_n/K) \simeq \mathbb{Z}/p^n\mathbb{Z}$ . One can consider

$$H_{\text{Sel}}^1(K_n, E_{p^\infty}) := \varinjlim_m H_{\text{Sel}}^1(K_n, E_{p^m}),$$

where the transition maps are induced by the inclusions  $E_{p^m} \hookrightarrow E_{p^{m+1}}$ . We now define

$$H_{\text{Sel}}^1(K_\infty, E_{p^\infty}) := \varinjlim_n H_{\text{Sel}}^1(K_n, E_{p^\infty}),$$

where the transition maps are simply restrictions. Observe that since we are assuming  $\text{Gal}(\mathbb{Q}(E_p)/\mathbb{Q})$  is not solvable, the transition maps in both of the above direct limits are injective. Since

$$\text{III}(K_n, E)_{p^\infty} = \varinjlim_m \text{III}(K_n, E)_{p^m} \quad \text{and} \quad \text{III}(K_\infty, E)_{p^\infty} = \varinjlim_n \text{III}(K_n, E)_{p^\infty},$$

the exactness of the sequence (1) implies that the sequence

$$0 \rightarrow E(\mathbb{K}_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow H_{\text{Sel}}^1(\mathbb{K}_\infty, E_{p^\infty}) \rightarrow \text{III}(\mathbb{K}_\infty, E)_{p^\infty} \rightarrow 0$$

is also exact.

Let us now choose a strictly increasing sequence of natural numbers  $\{m_n\}$  such that  $m_n \geq n$  and  $E(\mathbb{K}_{\lambda_n})_{p^\infty} \subseteq E_{p^{m_n}}$  for all primes  $\lambda_n \subset K_n$  which divide  $N$ , where  $\mathbb{K}_{\lambda_n}$  denotes the completion of  $K_n$  at  $\lambda_n$ . One can verify that

$$H_{\text{Sel}}^1(\mathbb{K}_\infty, E_{p^\infty}) = \varinjlim_n H_{\text{Sel}}^1(K_n, E_{p^{m_n}}).$$

For any finite set of rational primes  $\mathbb{Q}$ , we can consider the group

$$H_{\text{Sel}_{p \cup \mathbb{Q}}}^1(K_n, E_{p^{m_n}}) := \ker \left[ H^1(K_n, E_{p^{m_n}}) \rightarrow \prod_{\lambda_n \nmid \ell \in p \cup \mathbb{Q}} H^1(K_{\lambda_n}, E) \right],$$

where  $\lambda_n$  denotes primes of  $K_n$  and  $K_{\lambda_n}$  is the completion of  $K_n$  at  $\lambda_n$ . Notice that  $H_{\text{Sel}}^1(K_n, E_{p^{m_n}}) \subseteq H_{\text{Sel}_{p \cup \mathbb{Q}}}^1(K_n, E_{p^{m_n}})$ . Set

$$R_n := \mathbb{Z}/p^{m_n}\mathbb{Z}[\text{Gal}(K_n/K)]$$

and observe that  $H_{\text{Sel}_{p \cup \mathbb{Q}}}^1(K_n, E_{p^{m_n}})$  can be viewed as an  $R_n$ -module.

Let  $n' \geq n$ . The assumption that  $\text{Gal}(\mathbb{Q}(E_p)/\mathbb{Q})$  is not solvable implies that the restriction map

$$H^1(K_n, E_{p^{mn}}) \rightarrow H^1(K_{n'}, E_{p^{mn}}),$$

as well as the map

$$H^1(K_{n'}, E_{p^{mn}}) \rightarrow H^1(K_{n'}, E_{p^{m_{n'}}})$$

induced by the inclusion  $E_{p^{mn}} \hookrightarrow E_{p^{m_{n'}}}$ , are both injective. By composing the above maps, we obtain the injection

$$H^1(K_n, E_{p^{mn}}) \hookrightarrow H^1(K_{n'}, E_{p^{m_{n'}}}). \quad (2)$$

LEMMA 1.1. *The map (2) induces an isomorphism between the following  $R_n$ -modules:*

$$H_{\text{Sel}_{p \cup \mathbb{Q}}}^1(K_n, E_{p^{mn}}) \simeq H_{\text{Sel}_{p \cup \mathbb{Q}}}^1(K_{n'}, E_{p^{m_{n'}}})[g^{p^n} - 1, p^{mn}],$$

where  $\text{Gal}(K_\infty/K_n) = \langle g^{p^n} \rangle$  and  $n' \geq n$ .

*Proof.* The restriction map induces the isomorphism

$$H^1(K_n, E_{p^{m_{n'}}}) \simeq H^1(K_{n'}, E_{p^{m_{n'}}})^{\text{Gal}(K_\infty/K_n)} = H^1(K_{n'}, E_{p^{m_{n'}}})[g^{p^n} - 1].$$

Since  $H^1(K_n, E_{p^{mn}}) \simeq H^1(K_n, E_{p^{m_{n'}}})[p^{mn}]$ , it follows that

$$H^1(K_n, E_{p^{mn}}) \simeq H^1(K_{n'}, E_{p^{m_{n'}}})[g^{p^n} - 1, p^{mn}]$$

under the map (2). It is clear that

$$H_{\text{Sel}_{p \cup \mathbb{Q}}}^1(K_n, E_{p^{mn}}) \hookrightarrow H_{\text{Sel}_{p \cup \mathbb{Q}}}^1(K_{n'}, E_{p^{m_{n'}}})[g^{p^n} - 1, p^{mn}].$$

We will now show that the above map is surjective. Let  $\lambda_n$  be a prime of  $K_n$  and  $\lambda_{n'}$  a prime of  $K_{n'}$  that divides  $\lambda_n$ . We will assume that  $\lambda_n$  does not divide any of the primes in  $\{p\} \cup \mathbb{Q}_{k_{n'}}$ .

If  $\lambda_{n'}$  is a prime of good reduction, then the image of

$$H_{\text{Sel}_{p \cup \mathbb{Q}}}^1(K_{n'}, E_{p^{m_{n'}}})[g^{p^n} - 1, p^{mn}] \rightarrow H^1(K_{\lambda_{n'}}, E_{p^{m_{n'}}})$$

lies in  $H^1(K_{\lambda_{n'}}^{\text{unr}}/K_{\lambda_{n'}}, E_{p^{m_{n'}}})[g^{p^n} - 1, p^{mn}]$ ; here  $K_{\lambda_{n'}}$  denotes the completion of  $K_{n'}$  at  $\lambda_{n'}$ ,  $K_{\lambda_{n'}}^{\text{unr}}$  denotes its maximal unramified extension, and  $g^{p^n}$  generates  $\text{Gal}(K_{\lambda_{n'}}/K_{\lambda_n})$ . Since  $K_{\lambda_{n'}}/K_{\lambda_n}$  is unramified, the preimage of  $H^1(K_{\lambda_{n'}}^{\text{unr}}/K_{\lambda_{n'}}, E_{p^{m_{n'}}})[g^{p^n} - 1]$  under the restriction map

$$H^1(K_{\lambda_n}, E_{p^{m_{n'}}}) \rightarrow H^1(K_{\lambda_{n'}}, E_{p^{m_{n'}}})$$

lies in  $H^1(K_{\lambda_n}^{\text{unr}}/K_{\lambda_n}, E_{p^{m_{n'}}})$ . Finally, since

$$H^1(K_{\lambda_n}^{\text{unr}}/K_{\lambda_n}, E_{p^{m_{n'}}})[p^{mn}] = H^1(K_{\lambda_n}^{\text{unr}}/K_{\lambda_n}, E_{p^{mn}}),$$

we see that the image of

$$H_{\text{Sel}_{p \cup \mathbb{Q}_{k_{n'}}}}^1(K_{n'}, E_{p^{m_{n'}}})[g^{p^n} - 1, p^{mn}] \rightarrow H^1(K_{\lambda_n}, E_{p^{m_{n'}}})$$

lies in  $H^1(K_{\lambda_n}^{\text{unr}}/K_{\lambda_n}, E_{p^{mn}})$ .

If  $\lambda_{n'}$  is a prime of bad reduction, then the image of

$$H_{\text{Sel}_{p \cup \mathbb{Q}}}^1(K_{n'}, E_{p^{m_{n'}}})[g^{p^n} - 1, p^{mn}] \rightarrow H^1(K_{\lambda_{n'}}, E_{p^{m_{n'}}})$$

lies in the image of

$$(E(K_{\lambda_{n'}})/p^{m_{n'}})[g^{p^n} - 1, p^{mn}] \rightarrow H^1(K_{\lambda_{n'}}, E_{p^{m_{n'}}}).$$

By our choice of the sequence  $m_n$  and [ÇW08, Lemma 2.1.3], we know that

$$\mathrm{E}(\mathrm{K}_{\lambda_{n'}})/p^{m_{n'}} \simeq \mathrm{E}(\mathrm{K}_{\lambda_{n'}})_{p^{m_{n'}}} \quad \text{and} \quad \mathrm{E}(\mathrm{K}_{\lambda_n})/p^{m_n} \simeq \mathrm{E}(\mathrm{K}_{\lambda_n})_{p^{m_n}}.$$

It then follows that

$$(\mathrm{E}(\mathrm{K}_{\lambda_{n'}})/p^{m_{n'}})[g^{p^n} - 1, p^{m_n}] \simeq \mathrm{E}(\mathrm{K}_{\lambda_{n'}})_{p^{m_{n'}}}[g^{p^n} - 1, p^{m_n}] = \mathrm{E}(\mathrm{K}_{\lambda_n})_{p^{m_n}} \simeq \mathrm{E}(\mathrm{K}_{\lambda_n})/p^{m_n}.$$

This concludes the proof that the preimage of  $\mathrm{H}_{\mathrm{Sel}_{p \cup \mathbb{Q}}}^1(\mathrm{K}_{n'}, \mathrm{E}_{p^{m_{n'}}})[g^{p^n} - 1, p^{m_n}]$  under the map (2) is  $\mathrm{H}_{\mathrm{Sel}_{p \cup \mathbb{Q}}}^1(\mathrm{K}_n, \mathrm{E}_{p^{m_n}})$ .  $\square$

Let  $\{\mathbb{Q}_n \mid n \in \mathbb{N}\}$  be a sequence of sets of rational primes such that:

- (i)  $q \in \mathbb{Q}_n$  is inert in  $\mathrm{K}/\mathbb{Q}$ ;
- (ii)  $q \in \mathbb{Q}_n$  is prime to  $p\mathbb{N}$ ;
- (iii)  $\mathrm{E}(\mathrm{K}_q)_{p^\infty} = \mathrm{E}(\overline{\mathrm{K}_q})_{p^{m_n}}$ , where  $\mathrm{K}_q$  denotes the completion of  $\mathrm{K}$  at the prime of  $\mathrm{K}$  above  $q$ ;
- (iv)  $\mathrm{H}_{\mathrm{Sel}}^1(\mathrm{K}, \mathrm{E}_{p^{m_n}}) \hookrightarrow \prod_{q \in \mathbb{Q}_n} \mathrm{H}^1(\mathrm{K}_q, \mathrm{E}_{p^{m_n}})$ ;
- (v) the set  $\mathbb{Q}_n$  is finite and its size does not depend on  $n$ .

By [ÇW08, Proposition 2.6.3], all the  $\mathrm{R}_n$ -modules in the set

$$\{\mathrm{H}_{\mathrm{Sel}_{p \cup \mathbb{Q}_k}}^1(\mathrm{K}_n, \mathrm{E}_{p^{m_n}}) \mid k \geq n\}$$

have the same size. This implies that we can find a strictly increasing sequence  $\{k_n \in \mathbb{N} \mid n \in \mathbb{N}\}$  such that

$$\mathrm{H}_{\mathrm{Sel}_{p \cup \mathbb{Q}_{k_n}}}^1(\mathrm{K}_n, \mathrm{E}_{p^{m_n}}) \simeq \mathrm{H}_{\mathrm{Sel}_{p \cup \mathbb{Q}_{k_{n'}}}}^1(\mathrm{K}_n, \mathrm{E}_{p^{m_n}})$$

as  $\mathrm{R}_n$ -modules for all  $n' \geq n$ . Moreover, from Lemma 1.1 we know that

$$\mathrm{H}_{\mathrm{Sel}_{p \cup \mathbb{Q}_{k_{n'}}}}^1(\mathrm{K}_n, \mathrm{E}_{p^{m_n}}) \simeq \mathrm{H}_{\mathrm{Sel}_{p \cup \mathbb{Q}_{k_{n'}}}}^1(\mathrm{K}_{n'}, \mathrm{E}_{p^{m_{n'}}})[g^{p^n} - 1, p^{m_n}].$$

Consequently, even if the  $\mathrm{R}_n$ -modules  $\mathrm{H}_{\mathrm{Sel}_{p \cup \mathbb{Q}_{k_n}}}^1(\mathrm{K}_n, \mathrm{E}_{p^{m_n}})$  are not naturally related as  $n$  grows, we have that

$$\mathrm{H}_{\mathrm{Sel}_{p \cup \mathbb{Q}_{k_n}}}^1(\mathrm{K}_n, \mathrm{E}_{p^{m_n}}) \simeq \mathrm{H}_{\mathrm{Sel}_{p \cup \mathbb{Q}_{k_{n+1}}}}^1(\mathrm{K}_n, \mathrm{E}_{p^{m_n}}) \simeq \mathrm{H}_{\mathrm{Sel}_{p \cup \mathbb{Q}_{k_{n+1}}}}^1(\mathrm{K}_{n+1}, \mathrm{E}_{p^{m_{n+1}}})[g^{p^n} - 1, p^{m_n}],$$

where the first isomorphism is formal while the second is induced by the map (2). It follows that we can now fix maps

$$i_n : \mathrm{H}_{\mathrm{Sel}_{p \cup \mathbb{Q}_{k_n}}}^1(\mathrm{K}_n, \mathrm{E}_{p^{m_n}}) \rightarrow \mathrm{H}_{\mathrm{Sel}_{p \cup \mathbb{Q}_{k_{n+1}}}}^1(\mathrm{K}_{n+1}, \mathrm{E}_{p^{m_{n+1}}})$$

for every  $n \in \mathbb{N}$ , and we observe that all these maps are injective. Using  $i_n$  as transition maps, we construct the direct limit

$$\mathcal{M}_s := \varinjlim_n \mathrm{H}_{\mathrm{Sel}_{p \cup \mathbb{Q}_{k_n}}}^1(\mathrm{K}_n, \mathrm{E}_{p^{m_n}}).$$

The following theorem describes the structure of  $\mathcal{M}_s$  as a  $\Lambda$ -module.

**THEOREM 1.2** (Theorem 2.6.4 in [ÇW08]). *The  $\Lambda$ -module  $\widehat{\mathcal{M}}_s$  is isomorphic to  $\Lambda^{2t+2}$ , where  $t = \#\mathbb{Q}_{k_n}$ .*

Observe that for every  $n \in \mathbb{N}$  and any  $n' \geq n$  there is a noncanonical isomorphism

$$\mathrm{H}_{\mathrm{Sel}_{p \cup \mathbb{Q}_{k_{n'}}}}^1(\mathrm{K}_n, \mathrm{E}_{p^{m_n}}) \simeq \mathrm{H}_{\mathrm{Sel}_{p \cup \mathbb{Q}_{k_n}}}^1(\mathrm{K}_n, \mathrm{E}_{p^{m_n}})$$

and that the map

$$H_{\text{Sel}_{p \cup \mathbb{Q}_{k_n}}}^1(K_n, E_{p^{m_n}}) \rightarrow \mathcal{M}_s$$

is injective with image contained in  $\mathcal{M}_s[g^{p^n} - 1, p^{m_n}]$ . The composition therefore determines an injection

$$H_{\text{Sel}_{p \cup \mathbb{Q}_{k_{n'}}}}^1(K_n, E_{p^{m_n}}) \rightarrow \mathcal{M}_s[g^{p^n} - 1, p^{m_n}].$$

In addition, by [CW08, Proposition 2.6.3], we know that

$$\#H_{\text{Sel}_{p \cup \mathbb{Q}_{k_{n'}}}}^1(K_n, E_{p^{m_n}}) = \#(\mathbb{R}_n^{2t+2}) \quad \text{for all } n' \geq n.$$

This implies that

$$H_{\text{Sel}_{p \cup \mathbb{Q}_{k_{n'}}}}^1(K_n, E_{p^{m_n}}) \simeq \mathcal{M}_s[g^{p^n} - 1, p^{m_n}]$$

and, consequently, that

$$H_{\text{Sel}_{p \cup \mathbb{Q}_{k_{n'}}}}^1(K_n, E_{p^{m_n}}) \simeq \mathbb{R}_n^{2t+2} \quad \text{for every } n' \geq n.$$

Let us now consider the maps

$$H_{\text{Sel}_{p \cup \mathbb{Q}_{k_{n'}}}}^1(K_n, E_{p^{m_n}}) \rightarrow \prod_{q \in \mathbb{Q}_{k_{n'}}} H^1(K_n(q), E)_{p^{m_n}}, \quad (3)$$

where  $n' \geq n$  and  $H^1(K_n(q), E)_{p^{m_n}} := \prod_{q_n | q} H^1(K_{q_n}, E)_{p^{m_n}}$ , with  $q_n$  denoting primes of  $K_n$  above  $q$  and  $K_{q_n}$  denoting the completion of  $K_n$  at  $q_n$ . Notice that the kernel of the map (3) is  $H_{\text{Sel}_p}^1(K_n, E_{p^{m_n}})$  and, as in Lemma 1.1, one can see that

$$H_{\text{Sel}_p}^1(K_{n'}, E_{p^{m_{n'}}})[g^{p^n} - 1, p^{m_n}] \simeq H_{\text{Sel}_p}^1(K_n, E_{p^{m_n}}) \quad \text{for all } n' \geq n.$$

The first three properties of primes  $q \in \mathbb{Q}_n$  imply that (see [Ber95, Corollary 6])

$$H^1(K_n(q), E)_{p^{m_n}} \simeq \mathbb{R}_n^2.$$

Thus, the maps (3) can be viewed as maps between formal  $\mathbb{R}_n$ -modules

$$\theta_{n,n'} : \mathbb{R}_n^{2t+2} \rightarrow \mathbb{R}_n^{2t}.$$

Since for every  $n$  we have infinitely many maps  $\mathbb{R}_n^{2t+2} \rightarrow \mathbb{R}_n^{2t}$ , it follows that infinitely many of them are identical. This allows us to assume (by switching to a subsequence of the sequence  $k_n$  if necessary) that

$$\theta_{n,n'} = \theta_{n,n} \quad \text{for all } n' \geq n.$$

We now view  $\mathbb{R}_n$  as the  $\Lambda$ -module  $\hat{\Lambda}[g^{p^n} - 1, p^{m_n}]$ . It is then easy to see that  $\mathbb{R}_n \subseteq \mathbb{R}_{n+1}$ . By using Lemma 1.1, the fact that

$$H^1(K_{n+1}(q), E)_{p^{m_{n+1}}}[g^{p^n} - 1, p^{m_n}] = H^1(K_n(q), E)_{p^{m_n}}$$

and the commutative diagram

$$\begin{array}{ccc} H_{\text{Sel}_{p \cup \mathbb{Q}_{k_{n+1}}} }^1(K_{n+1}, E_{p^{m_{n+1}}}) & \longrightarrow & \prod_{q \in \mathbb{Q}_{k_{n+1}}} H^1(K_{n+1}(q), E)_{p^{m_{n+1}}} \\ \uparrow & & \uparrow \\ H_{\text{Sel}_{p \cup \mathbb{Q}_{k_{n+1}}} }^1(K_n, E_{p^{m_n}}) & \longrightarrow & \prod_{q \in \mathbb{Q}_{k_{n+1}}} H^1(K_n(q), E)_{p^{m_n}} \end{array}$$

we see that the diagram

$$\begin{array}{ccc} R_{n+1}^{2t+2} & \xrightarrow{\theta_{n+1,n+1}} & R_{n+1}^{2t} \\ \uparrow & & \uparrow \\ R_n^{2t+2} & \xrightarrow{\theta_{n,n+1}} & R_n^{2t} \end{array}$$

commutes. Since  $\theta_{n,n+1} = \theta_{n,n}$ , we can now consider the  $\Lambda$ -module map

$$\theta : \hat{\Lambda}^{2t+2} \rightarrow \hat{\Lambda}^{2t}, \quad (4)$$

where the restriction of  $\theta$  to  $R_n^{2t+2}$  equals  $\theta_{n,n}$ .

Notice that the kernel of the map (3) is  $H_{\text{Sel}_p}^1(K_n, E_{p^{m_n}})$ , which is equivalent to saying that

$$\ker \theta_{n,n} \simeq H_{\text{Sel}_p}^1(K_n, E_{p^{m_n}}).$$

Consequently, the kernel of the map  $\theta$  is a direct limit of  $H_{\text{Sel}_p}^1(K_n, E_{p^{m_n}})$ , where the transition maps are injective but not necessarily the natural ones.

**PROPOSITION 1.3.** *The  $\Lambda$ -corank of  $H_{\text{Sel}_p}^1(K_\infty, E_{p^\infty})$  is equal to the  $\Lambda$ -corank of the kernel of  $\theta$ .*

*Proof.* As in Lemma 1.1, we can show that

$$H_{\text{Sel}_p}^1(K_{n'}, E_{p^{m_{n'}}})[g^{p^n} - 1, p^{m_n}] \simeq H_{\text{Sel}_p}^1(K_n, E_{p^{m_n}}) \quad \text{for all } n' \geq n. \quad (5)$$

On the one hand, since  $H_{\text{Sel}_p}^1(K_\infty, E_{p^\infty}) = \varinjlim_{n'} H_{\text{Sel}_p}^1(K_{n'}, E_{p^{m_{n'}}})$ , we have

$$H_{\text{Sel}_p}^1(K_\infty, E_{p^\infty})[g^{p^n} - 1, p^{m_n}] \simeq H_{\text{Sel}_p}^1(K_n, E_{p^{m_n}}).$$

On the other hand, (5) and the fact that the transition maps used in viewing  $\ker \theta$  as a direct limit of  $H_{\text{Sel}_p}^1(K_n, E_{p^{m_n}})$  are injective together imply that

$$\ker \theta [g^{p^n} - 1, p^{m_n}] \simeq H_{\text{Sel}_p}^1(K_n, E_{p^{m_n}}).$$

So we have that

$$\ker \theta [g^{p^n} - 1, p^{m_n}] \simeq H_{\text{Sel}_p}^1(K_\infty, E_{p^\infty})[g^{p^n} - 1, p^{m_n}],$$

which implies that the  $\Lambda$ -corank of  $H_{\text{Sel}_p}^1(K_\infty, E_{p^\infty})$  equals that of the kernel of  $\theta$ .  $\square$

## 2. Heegner points and Kolyvagin classes

**2.1** We fix a parametrization  $\pi : X_0(N) \rightarrow E$  which maps the cusp at  $\infty$  to the origin of  $E$  (see [BCDT01] and [Wil95]). Let  $\mathcal{O}_K$  be the ring of integers of  $K$ . Since we have assumed that the primes dividing  $N$  (the conductor of  $E$ ) split in  $K/\mathbb{Q}$ , we can choose an ideal  $\mathcal{N}$  such that  $\mathcal{O}_K/\mathcal{N} \simeq \mathbb{Z}/N\mathbb{Z}$ . For any positive integer  $f$  prime to  $N$ , we can consider  $x_f = (\mathbb{C}/\mathcal{O}_f, \mathbb{C}/\mathcal{N}_f) \in X_0(N)$ , where  $\mathcal{O}_f$  denotes the order of  $K$  of conductor  $f$  and  $\mathcal{N}_f = \mathcal{N} \cap \mathcal{O}_f$ . We define the Heegner point by  $y_f = \pi(x_f)$ . The Heegner point  $y_f$  is defined over  $K[\mathfrak{f}]$ , the ring class field of  $K$  of conductor  $\mathfrak{f}$ .

Let  $\tilde{K}_\infty = \bigcup_{n \geq 1} K[p^n]$ . Then  $\text{Gal}(\tilde{K}_\infty/K)$  is isomorphic to  $\mathbb{Z}_p \times \Delta$ , where  $\Delta$  is a finite abelian group. The unique  $\mathbb{Z}_p$ -extension contained in  $\tilde{K}_\infty$  is the anticyclotomic  $\mathbb{Z}_p$ -extension  $K_\infty$ . Denote by  $K[p^{k(n)}]$  the minimal ring class field of  $p$ -power conductor that contains  $K_n$ , the subextension

of  $K_\infty$  of degree  $p^n$  over  $K$ . We then define  $\alpha_n \in E(K_n)$  to be the trace of  $y_{p^{k(n)}}$  from  $K[p^{k(n)}]$  to  $K_n$ . Perrin-Riou [Per87, § 3.3, Lemma 2] has shown that

$$a_p y_{p^{n+1}} = y_{p^n} + \mathrm{tr}_{K[p^{n+2}]/K[p^{n+1}]} y_{p^{n+2}} \quad \text{for } n \geq 0.$$

Since we are assuming that  $\mathrm{Gal}(\mathbb{Q}(E_p)/\mathbb{Q})$  is not solvable, it follows that  $p \geq 5$ ; in conjunction with the fact that  $E$  has supersingular reduction at  $p$ , this implies that  $a_p = 0$ . We can therefore deduce that

$$\mathrm{tr}_{K_{n+2}/K_n} \alpha_{n+2} = -\alpha_n \tag{6}$$

for all  $n \geq k_0 := \max\{n \in \mathbb{N} \mid K_n \subseteq K[1]\}$ .

For any  $n' \geq n$ , let  $R_{n'}\alpha_n$  denote the  $R_{n'}$ -submodule of  $H^1(K_{n'}, E_{p^{m_{n'}}})$  generated by the image of  $\alpha_n$  under the map

$$E(K_{n'}) \rightarrow H^1(K_{n'}, E_{p^{m_{n'}}}).$$

Since the group  $\mathrm{Gal}(\mathbb{Q}(E_p)/\mathbb{Q})$  is not solvable, the map

$$H^1(K_n, E_{p^{m_n}}) \rightarrow H^1(K_{n'}, E_{p^{m_{n'}}}) \tag{7}$$

is injective and induces the isomorphism

$$R_n\alpha_n \simeq R_{n'}(p^{m_{n'}-m_n}\alpha_n).$$

By using

$$R_{n'}(p^{m_{n'}-m_n}\alpha_n) \subseteq R_{n'}\alpha_n \subseteq H^1(K_{n'}, E_{p^{m_{n'}}}),$$

we see that the map (7) induces the injective homomorphism

$$R_n\alpha_n \hookrightarrow R_{n'}\alpha_n.$$

Moreover, the relations (6) imply that

$$R_{n+2k}\alpha_n \subseteq R_{n+2k}\alpha_{n+2k} \subseteq H_{\mathrm{Sel}}^1(K_{n+2k}, E_{p^{m_{n+2k}}}) \subseteq H_{\mathrm{Sel}}^1(K_\infty, E_{p^\infty}),$$

where  $k$  is any positive integer. Hence, we have the following maps:

$$R_{2n+1}\alpha_{2n} \hookrightarrow R_{2n'+1}\alpha_{2n'} \quad \text{and} \quad R_{2n+1}\alpha_{2n+1} \hookrightarrow R_{2n'+1}\alpha_{2n'+1},$$

which can be used as transition maps in defining the direct limits

$$\varinjlim_n R_{2n+1}\alpha_{2n} \quad \text{and} \quad \varinjlim_n R_{2n+1}\alpha_{2n+1}.$$

Since the transition maps of the above direct limits are simply restrictions of the maps (7), these direct limits are submodules of  $H_{\mathrm{Sel}}^1(K_\infty, E_{p^\infty})$ .

**PROPOSITION 2.1.** *The  $\Lambda$ -modules  $\varinjlim_n R_{2n+1}\alpha_{2n}$  and  $\varinjlim_n R_{2n+1}\alpha_{2n+1}$  have nontrivial coranks, and together they give rise to a submodule of  $H_{\mathrm{Sel}_p}^1(K_\infty, E_{p^\infty})$  of corank greater than or equal to two.*

*Remark 2.2.* Observe that, while the statement of this proposition is the same as that of [CW08, Lemma 2.6.5], in this case we do not assume that  $K_\infty/K$  is totally ramified at the primes above  $p$ .

*Proof.* Cornut [Cor02] and Vatsal [Vat03] have shown that all but finitely many of the Heegner points are nontorsion. Using this result, one can show (see [CW08, Proposition 2.5.1]) that the  $\Lambda$ -modules  $\varinjlim_n R_{2n+1}\alpha_{2n}$  and  $\varinjlim_n R_{2n+1}\alpha_{2n+1}$  have nontrivial coranks. It then follows that we can restrict our attention to the case where each of these submodules of  $H_{\mathrm{Sel}_p}^1(K_\infty, E_{p^\infty})$  has

$\Lambda$ -corank one. In this case, we will consider the restrictions of the submodules at primes above  $p$  and analyze their image in the local cohomology group.

Let  $\wp$  be a prime of  $K$  above  $p$ ,  $\wp_n$  a prime of  $K_n$  dividing  $\wp$ ,  $K_\wp$  the completion of  $K$  at  $\wp$ , and  $K_{\wp_n}$  the completion of  $K_n$  at  $\wp_n$ . Following Kobayashi [Kob03], we define the following subgroups of  $E(K_{\wp_n})$ :

$$\begin{aligned} E^+(K_{\wp_n}) &:= \{x \in E(K_{\wp_n}) \mid \text{tr}_{K_{\wp_n}/K_{\wp_{m+1}}}(x) \in E(K_{\wp_m}) \text{ for all } k_0 \leq m < n, m \text{ even}\}, \\ E^-(K_{\wp_n}) &:= \{x \in E(K_{\wp_n}) \mid \text{tr}_{K_{\wp_n}/K_{\wp_{m+1}}}(x) \in E(K_{\wp_m}) \text{ for all } k_0 \leq m < n, m \text{ odd}\}. \end{aligned}$$

Observe that  $\text{res}_{\wp_{2n}} \alpha_{2n} \in E^+(K_{\wp_{2n}})$  and  $\text{res}_{\wp_{2n+1}} \alpha_{2n+1} \in E^-(K_{\wp_{2n+1}})$ , where

$$\text{res}_{\wp_n} : E(K_n) \rightarrow E(K_{\wp_n}).$$

Since the subgroups  $E^\pm(K_{\wp_n})$  are closed under the action of  $\text{Gal}(K_{\wp_n}/K_\wp)$ , they can be viewed as  $\mathbb{Z}_p[\text{Gal}(K_{\wp_n}/K_\wp)]$ -modules. Our aim now is to show that  $\varinjlim_n E^\pm(K_{\wp_n}) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ , viewed as modules over  $\Lambda = \mathbb{Z}_p[\text{Gal}(K_{\wp_\infty}/K_\wp)]$ , have corank one.

We know that the  $\mathbb{Z}_p$ -rank of  $E(K_{\wp_n})$  equals that of  $\mathcal{O}_{\wp_n}$ , the ring of integers of  $K_{\wp_n}$ . Since  $E(K_{\wp_n})/(E^+(K_{\wp_n}) + E^-(K_{\wp_n}))$  is annihilated simultaneously by

$$\prod_{k_0 < 2m \leq n} \text{tr}_{K_{\wp_{2m}}/K_{\wp_{2m-1}}} \quad \text{and} \quad \prod_{k_0 < 2m+1 \leq n} \text{tr}_{K_{\wp_{2m+1}}/K_{\wp_{2m}}},$$

it follows that

$$p^n E(K_{\wp_n}) \subseteq E^-(K_{\wp_n}) + E^+(K_{\wp_n}).$$

Moreover,  $E^+(K_{\wp_{2m+1}}) \subseteq E(K_{\wp_{2m}})$  and  $E^-(K_{\wp_{2m}}) \subseteq E(K_{\wp_{2m-1}})$  for all  $m \geq k_0$ . Consequently, we can deduce the following facts about the  $\mathbb{Z}_p$ -ranks of  $E^\pm(K_{\wp_n})$ :

$$\begin{aligned} \text{rank}_{\mathbb{Z}_p} E^+(K_{\wp_n}) &= \text{rank}_{\mathbb{Z}_p} \mathcal{O}_{\wp_{k_0}} + \sum_{k_0 < 2m \leq n} (\text{rank}_{\mathbb{Z}_p} \mathcal{O}_{\wp_{2m}} - \text{rank}_{\mathbb{Z}_p} \mathcal{O}_{\wp_{2m-1}}), \\ \text{rank}_{\mathbb{Z}_p} E^-(K_{\wp_n}) &= \text{rank}_{\mathbb{Z}_p} \mathcal{O}_{\wp_{k_0}} + \sum_{k_0 < 2m+1 \leq n} (\text{rank}_{\mathbb{Z}_p} \mathcal{O}_{\wp_{2m+1}} - \text{rank}_{\mathbb{Z}_p} \mathcal{O}_{\wp_{2m}}). \end{aligned}$$

Let  $r_0 = \min\{n \in \mathbb{N} \mid \alpha_{2n} \notin E(K_{\wp_{2n}})_{\text{tors}}, 2n \geq k_0\}$ . Then, for some  $f_0(g)$  dividing  $g^{k_0} - 1$ , we have that

$$f_0(g) \prod_{r_0 < r \leq m} \text{tr}_{K_{\wp_{2r}}/K_{\wp_{2r-1}}}$$

is a minimal annihilator  $\mathbb{Z}_p[\text{Gal}(K_{\wp_{2m}}/K_\wp)] \text{res}_{\wp_{2m}} \alpha_{2m} \subseteq E^+(K_{\wp_{2m}})$  and  $r_0$  is by definition independent of  $m$ . This implies that the difference between the  $\mathbb{Z}_p$ -rank of  $E^+(K_{\wp_{2m}})$  and the  $\mathbb{Z}_p$ -rank of its submodule  $\mathbb{Z}_p[\text{Gal}(K_{\wp_{2m}}/K_\wp)] \text{res}_{\wp_{2m}} \alpha_{2m}$  is bounded independently of  $m$ . One can draw the same conclusion about the difference between the  $\mathbb{Z}_p$ -ranks of  $E^-(K_{\wp_{2m+1}})$  and  $\mathbb{Z}_p[\text{Gal}(K_{\wp_{2m+1}}/K_\wp)] \text{res}_{\wp_{2m+1}} \alpha_{2m+1}$ . It then follows that

$$\text{corank}_\Lambda \varinjlim_n \mathbb{Z}_p[\text{Gal}(K_{\wp_{2n}}/K_\wp)] \text{res}_{\wp_{2n}} \alpha_{2n} \otimes \mathbb{Q}_p/\mathbb{Z}_p = \text{corank}_\Lambda \varinjlim_n E^+(K_{\wp_n}) \otimes \mathbb{Q}_p/\mathbb{Z}_p, \quad (8)$$

$$\text{corank}_\Lambda \varinjlim_n \mathbb{Z}_p[\text{Gal}(K_{\wp_{2n+1}}/K_\wp)] \text{res}_{\wp_{2n+1}} \alpha_{2n+1} \otimes \mathbb{Q}_p/\mathbb{Z}_p = \text{corank}_\Lambda \varinjlim_n E^-(K_{\wp_n}) \otimes \mathbb{Q}_p/\mathbb{Z}_p. \quad (9)$$

Since  $\alpha_n$  is nontorsion for almost all  $n$ , the same holds for  $\text{res}_{\wp_n} \alpha_n$ . Consequently, as in [CW08, Proposition 2.5.1], one can show that the modules  $\varinjlim_n \mathbb{Z}_p[\text{Gal}(K_{\wp_{2n}}/K_\wp)] \text{res}_{\wp_{2n}} \alpha_{2n} \otimes \mathbb{Q}_p/\mathbb{Z}_p$  and  $\varinjlim_n \mathbb{Z}_p[\text{Gal}(K_{\wp_{2n+1}}/K_\wp)] \text{res}_{\wp_{2n+1}} \alpha_{2n+1} \otimes \mathbb{Q}_p/\mathbb{Z}_p$  have nontrivial  $\Lambda$ -coranks.

Moreover, using the fact that the maps

$$\varinjlim_n R_{2n+1} \alpha_{2n} \rightarrow \varinjlim_n \mathbb{Z}_p[\text{Gal}(K_{\wp_{2n}}/K_\wp)] \text{res}_{\wp_{2n}} \alpha_{2n} \otimes \mathbb{Q}_p/\mathbb{Z}_p$$

and

$$\varinjlim_n R_{2n+1} \alpha_{2n+1} \rightarrow \varinjlim_n \mathbb{Z}_p[\text{Gal}(K_{\wp_{2n+1}}/K_\wp)] \text{res}_{\wp_{2n+1}} \alpha_{2n+1} \otimes \mathbb{Q}_p/\mathbb{Z}_p$$

are surjective, together with our assumption that  $\varinjlim_n R_{2n+1} \alpha_{2n}$  and  $\varinjlim_n R_{2n+1} \alpha_{2n+1}$  have  $\Lambda$ -corank one, we deduce that

$$\text{corank}_\Lambda \varinjlim_n \mathbb{Z}_p[\text{Gal}(K_{\wp_{2n}}/K_\wp)] \text{res}_{\wp_{2n}} \alpha_{2n} \otimes \mathbb{Q}_p/\mathbb{Z}_p = 1,$$

$$\text{corank}_\Lambda \varinjlim_n \mathbb{Z}_p[\text{Gal}(K_{\wp_{2n+1}}/K_\wp)] \text{res}_{\wp_{2n+1}} \alpha_{2n+1} \otimes \mathbb{Q}_p/\mathbb{Z}_p = 1.$$

Hence, in view of (8) and (9), we have that

$$\text{corank}_\Lambda \varinjlim_n E^+(K_{\wp_n}) \otimes \mathbb{Q}_p/\mathbb{Z}_p = \text{corank}_\Lambda \varinjlim_n E^-(K_{\wp_n}) \otimes \mathbb{Q}_p/\mathbb{Z}_p = 1.$$

Consider the exact sequence

$$0 \rightarrow E^+(K_{\wp_n}) + E^-(K_{\wp_n}) \rightarrow E(K_{\wp_n}) \rightarrow E(K_{\wp_n})/(E^+(K_{\wp_n}) + E^-(K_{\wp_n})) \rightarrow 0.$$

The last term is annihilated by  $p^n$ . Moreover, since  $p$  is a prime of supersingular reduction and  $K_{\wp_n}/\mathbb{Q}_p$  is a cyclic Galois extension, we know that  $E(K_{\wp_n})_p = 0$ . Hence, by applying the snake lemma, we get

$$\begin{aligned} 0 &\rightarrow E(K_{\wp_n})/(E^+(K_{\wp_n}) + E^-(K_{\wp_n})) \\ &\rightarrow (E^+(K_{\wp_n}) + E^-(K_{\wp_n})) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow E(K_{\wp_n}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow 0. \end{aligned}$$

The fact that  $p$  is a supersingular prime also implies that

$$\varinjlim_n E(K_{\wp_n}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \simeq \varinjlim_n H^1(K_{\wp_n}, E_{p^\infty})$$

under the natural inclusion maps. In addition, we know that (see [Gre01, ch. 2])

$$\text{corank}_{\mathbb{Z}_p} H^1(K_{\wp_n}, E_{p^\infty}) = 2[K_{\wp_n} : K_\wp],$$

which implies that  $\varinjlim_n H^1(K_{\wp_n}, E_{p^\infty})$  has  $\Lambda$ -corank two. By the above we have that

$$\varinjlim_n E^+(K_{\wp_n}) \otimes \mathbb{Q}_p/\mathbb{Z}_p + \varinjlim_n E^-(K_{\wp_n}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow \varinjlim_n H^1(K_{\wp_n}, E_{p^\infty});$$

therefore the cokernel of

$$\begin{aligned} &\varinjlim_n \mathbb{Z}_p[\text{Gal}(K_{\wp_{2n}}/K_\wp)] \text{res}_{\wp_{2n}} \alpha_{2n} \otimes \mathbb{Q}_p/\mathbb{Z}_p \\ &+ \varinjlim_n \mathbb{Z}_p[\text{Gal}(K_{\wp_{2n+1}}/K_\wp)] \text{res}_{\wp_{2n+1}} \alpha_{2n+1} \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow \varinjlim_n H^1(K_{\wp_n}, E_{p^\infty}) \end{aligned}$$

is torsion over  $\Lambda$  and, consequently, the image of

$$\varinjlim_n R_{2n+1}\alpha_{2n} + \varinjlim_n R_{2n+1}\alpha_{2n+1} \quad \text{in} \quad \varinjlim_n H^1(K_{\wp_n}, E_{p^\infty})$$

has  $\Lambda$ -corank two. Thus we can now conclude that the Heegner points give rise to a submodule of  $H_{\text{Sel}_p}^1(K_\infty, E_{p^\infty})$  of  $\Lambda$ -corank greater than or equal to two.  $\square$

**2.2** Kolyvagin used Heegner points to construct cohomology classes whose ramification can be controlled. We will now describe a natural generalization of Kolyvagin's cohomology classes to ring class fields (following [BD90]). Let  $r$  be a squarefree product of primes  $\ell|r$  satisfying the following conditions:

- (i)  $\ell$  is relatively prime to  $p\text{ND}_K$ ;
- (ii)  $\tau \in \text{Frob}_\ell(K(E_{p^{m_{n'}}})/\mathbb{Q})$ , where  $\tau$  denotes complex conjugation.

Let  $k_0 \leq n \leq n'$ , and denote by  $K_n[r]$  the maximal subextension of  $K_n K[r]$  which is a  $p$ -primary extension of  $K_n$ . We now define  $\alpha_n(r)$  to be the trace of  $y_{r,p^{k(n)}}$  over  $K[rp^{k(n)}]/K_n[r]$ .

Let  $\mathcal{G}_{n,r} = \text{Gal}(K_n[r]/K_n[r] \cap K_n K[1])$  and  $\mathcal{G}_{n,\ell} = \text{Gal}(K_n[\ell]/K_n[\ell] \cap K_n K[1])$ . By class field theory,  $\mathcal{G}_{n,r} = \prod_{\ell|r} \mathcal{G}_{n,\ell}$  and  $\mathcal{G}_{n,\ell} \simeq \mathbb{Z}/p^{n_\ell}\mathbb{Z}$  for  $n_\ell = p^{\text{ord}_p(\ell+1)}$ . Consider  $D_\ell := \sum_{i=1}^{n_\ell} i\sigma_\ell^i \in \mathbb{Z}/p^{m_n}\mathbb{Z}[\mathcal{G}_{n,\ell}]$  and  $D_r := \prod_{\ell|r} D_\ell \in \mathbb{Z}/p^{m_n}\mathbb{Z}[\mathcal{G}_{n,r}]$  (with  $D_1 := 1$ ). One can then show that  $D_r \alpha_n(r)$  belongs to  $(E(K_n[r])/p^{m_n})^{\mathcal{G}_{n,r}}$  (see [BD90, Lemma 3.3]). It follows that

$$\text{tr}_{(K_n[r] \cap K_n K[1])/K_n} D_r \alpha_n(r) \in (E(K_n[r])/p^{m_n})^{\mathcal{G}_{n,r}},$$

where  $\mathcal{G}_{n,r} = \text{Gal}(K_n[r]/K_n)$ . We now consider the following commutative diagram.

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & H^1(K_n[r]/K_n, E)_{p^{m_n}} & & \\
 & & & & \text{Inf} \downarrow & & \\
 0 & \longrightarrow & E(K_n)/p^{m_n} E(K_n) & \xrightarrow{\phi} & H^1(K_n, E_{p^{m_n}}) & \longrightarrow & H^1(K_n, E)_{p^{m_n}} \longrightarrow 0 \\
 & & \downarrow & & \text{res} \downarrow \wr & & \text{res} \downarrow \\
 0 & \longrightarrow & (E(K_n[r])/p^{m_n})^{\mathcal{G}_{n,r}} & \xrightarrow{\phi_r} & H^1(K_n[r], E_{p^{m_n}})^{\mathcal{G}_{n,r}} & \longrightarrow & H^1(K_n[r], E)_{p^{m_n}}^{\mathcal{G}_{n,r}}
 \end{array} \tag{10}$$

Let  $c_n(r) \in H^1(K_n, E_{p^{m_n}})$  be such that

$$\phi_r(\text{tr}_{(K_n[r] \cap K_n K[1])/K_n} D_r \alpha_n(r)) = \text{res}(c_n(r)),$$

and let  $d_n(r)$  be the image of  $c_n(r)$  in  $H^1(K_n, E)_{p^{m_n}}$ . In particular,  $\text{res}(c_n(1)) = \phi_1(\alpha_n)$ . These generalized Kolyvagin cohomology classes have the following properties.

- (1) Let  $-\epsilon$  denote the sign of the functional equation of the L-function of  $E/\mathbb{Q}$ , and let  $f_r$  be the number of prime divisors of  $r$ . After extending  $\tau$  to a complex conjugation in  $\text{Gal}(K_\infty/\mathbb{Q})$ , we see that  $\tau$  acts on  $\alpha_n$  with  $\tau\alpha_n = \epsilon g^{i_{n,1}}\alpha_n + \beta_n$ , where  $\beta_n \in E(K_n)_{\text{tors}}$ ,  $g$  is a generator of  $\text{Gal}(K_\infty/K)$  and  $i_{n,1} \in \{0, \dots, p^n - 1\}$ . Moreover, the complex conjugation  $\tau$  acts on  $H^1(K_n, E_{p^{m_n}})$ , and we can deduce that  $\tau c_n(r) = \epsilon_r g^{i_{n,r}} c_n(r)$  where  $\epsilon_r = (-1)^{f_r} \epsilon$  and  $i_{n,r} \in \{0, \dots, p^n - 1\}$ .

- (2) If  $v$  is a rational prime which does not divide  $r$ , then  $d_n(r)_{v_n} = 0$  in  $H^1(K_{v_n}, E)_{p^{m_n}}$  for all primes  $v_n$  of  $K_n$  such that  $v_n|v$ .
- (3) Let  $H^1(K_n(\ell), E_{p^{m_n}}) := \prod_{\lambda_n|\ell} H^1(K_{\lambda_n}, E_{p^{m_n}})$  and  $H^1(K_n(\ell), E)_{p^{m_n}} := \prod_{\lambda_n|\ell} H^1(K_{\lambda_n}, E)_{p^{m_n}}$ . Define  $\text{res}_\ell$  and  $\text{res}_\ell$  to be the following localization maps:

$$\begin{aligned} \text{res}_\ell &: H^1(K_n, E_{p^{m_n}}) \rightarrow H^1(K_n(\ell), E_{p^{m_n}}), \\ \text{res}_\ell &: H^1(K_n, E)_{p^{m_n}} \rightarrow H^1(K_n(\ell), E)_{p^{m_n}}. \end{aligned}$$

We set  $E(K_n(\ell))/p^{m_n} := \prod_{\lambda_n|\ell} E(K_{\lambda_n})/p^{m_n}$ . Then if  $\ell|r$ , there exists a  $G_n$ -equivariant and  $\tau$ -antiequivariant isomorphism

$$\psi_\ell : H^1(K_n(\ell), E)_{p^{m_n}} \rightarrow E(K_n(\ell))/p^{m_n}$$

such that  $\psi_\ell(\text{res}_\ell(d_n(r))) = \text{res}_\ell(c_n(r/\ell))$ .

- (4) As in the case where  $r = 1$  (see § 2.1), Perrin-Riou [Per87, § 3.3, Lemma 2]) has shown that

$$a_p y_{rp^{n+1}} = y_{rp^n} + \text{tr}_{K[rp^{n+2}]/K[rp^{n+1}]} y_{rp^{n+2}}$$

for any  $n \geq 0$  and any  $r \in \mathbb{N}$  prime to  $p$ . Since  $a_p = 0$ , it follows that

$$y_{rp^n} = -\text{tr}_{K[rp^{n+2}]/K[rp^{n+1}]} y_{rp^{n+2}}. \quad (11)$$

Let  $R_n c_n(r)$  be the  $R_n$ -submodule of  $H^1(K_n, E_{p^{m_n}})$  generated by  $c_n(r)$ . Under the injective map

$$H^1(K_n, E_{p^{m_n}}) \rightarrow H^1(K_{n+2}, E_{p^{m_{n+2}}}),$$

$R_n c_n(r)$  can be viewed as a submodule of  $H^1(K_{n+2}, E_{p^{m_{n+2}}})$ . Moreover, by (11) we can then see that  $R_n c_n(r) \subseteq R_{n+2} c_{n+2}(r)$  and, consequently, that  $R_n d_n(r) \subseteq R_{n+2} d_{n+2}(r)$ .

By identifying  $R_{2n} \alpha_{2n}$  with its image under the injective map

$$H^1(K_{2n}, E_{p^{m_{2n}}}) \rightarrow H^1(K_{2n+1}, E_{p^{m_{2n+1}}}),$$

we now view  $R_{2n} \alpha_{2n} + R_{2n+1} \alpha_{2n+1}$  as an  $R_{2n+1}$ -submodule of  $H^1(K_{2n+1}, E_{p^{m_{2n+1}}})$ .

PROPOSITION 2.3. *For almost all  $n \in \mathbb{N}$ , there exists a set of rational primes*

$$Q_n = \{\ell_n(1), \dots, \ell_n(t)\}$$

satisfying the following properties:

- (i)  $\ell_n(i)$  is inert in  $K/\mathbb{Q}$ ;
- (ii)  $\ell_n(i)$  is prime to  $p\mathbb{N}$ ;
- (iii)  $E(K_\lambda)_{p^\infty} = E(\overline{K_\lambda})_{p^{m_n}}$  for all  $\lambda | \ell_n(i)$ , where  $K_\lambda$  denotes the completion of  $K$  at  $\lambda$ ;
- (iv)  $H_{\text{Sel}}^1(K, E_{p^{m_n}}) \hookrightarrow \prod_{i=1}^t H^1(K_{\lambda_n(i)}, E_{p^{m_n}})$ ;
- (v) the images of  $R_{2n} \alpha_{2n} + R_{2n+1} \alpha_{2n+1}$  under

$$\text{res}_{\ell_m(i)} : H^1(K_{2n+1}, E_{p^{m_{2n+1}}}) \rightarrow H^1(K_{2n+1}(\ell_m(i)), E_{p^{m_{2n+1}}})$$

are isomorphic as  $R_{2n+1}$ -modules for all  $m \geq 2n + 1$ ;

- (vi) the direct limits

$$\varinjlim_n \text{res}_{\ell_{2n+1}(i)} (R_{2n} \alpha_{2n} + R_{2n+1} \alpha_{2n+1}),$$

which will be defined using injective transition maps, have  $\Lambda$ -corank two for each  $i \in \{1, \dots, t\}$ .

*Proof.* Let  $L_n = K(E_{p^{m_n}})$  and  $\mathcal{G}_n = \text{Gal}(L_n/K)$ . Consider the exact sequence

$$0 \rightarrow H^1(\mathcal{G}_n, E_{p^{m_n}}) \rightarrow H^1(K, E_{p^{m_n}}) \xrightarrow{\text{res}} H^1(L_n, E_{p^{m_n}})^{\mathcal{G}_n}. \quad (12)$$

Since  $H^1(\mathcal{G}_n, E_{p^{m_n}}) = 0$  for all  $n$  [ÇW08, Proposition 1.3.1], the above diagram implies that

$$H^1(K, E_{p^{m_n}}) \hookrightarrow H^1(L_n, E_{p^{m_n}})^{\mathcal{G}_n} = \text{Hom}_{\mathcal{G}_n}(\text{Gal}(\overline{L}_n/L_n), E_{p^{m_n}}).$$

Let  $M_n$  be the splitting field over  $L_n$  of the finite subgroup  $H_{\text{Sel}}^1(K, E_{p^{m_n}})$  of  $H^1(L_n, E_{p^{m_n}})^{\mathcal{G}_n}$ . The complex conjugation  $\tau$  acts on  $\text{Gal}(M_n/L_n)$  and the  $+1$  eigenspace

$$\text{Gal}(M_n/L_n)^+ = \{(\tau h)^2 \mid h \in \text{Gal}(M_n/L_n)\}.$$

Fix  $\{h_n(1), \dots, h_n(t)\}$  to be a minimal set of generators of  $\text{Gal}(M_n/L_n)^+$ . One can easily see that  $t$  does not depend on  $n$ . We then choose primes  $\ell_n(i) \in \mathbb{Q}$  such that  $\tau h'_n(i) \in \text{Frob}_{\ell_n}(M_n/\mathbb{Q})$ , where  $h_n(i) = (\tau h'_n(i))^2$ . This choice ensures that the prime  $\ell_n(i)$  satisfies the first two required properties.

In [ÇW08, § 1.3.2] we showed that  $M_n$  and  $L_{n+1}$  are disjoint over  $L_n$ . We also know that the index of  $\text{Gal}(L_n/K)$  in  $\text{GL}(2, \mathbb{Z}/p^{m_n}\mathbb{Z})$  is finite and depends only on  $E$  and  $K$  (see [Ser72]). This implies that, for almost all  $n$ , we can extend each  $\tau h'_n(i)$  to an element of  $\text{Gal}(M_n K(E_{p^{m_{n+1}}})/\mathbb{Q})$  in such a way that the restriction of  $(\tau h'_n(i))^2$  to  $\text{Gal}(K(E_{p^{m_{n+1}}})/L_n)$  has no fixed points in  $E_{p^{m_{n+1}}}/E_{p^{m_n}}$ . Hence we have

$$E(K_\lambda)_{p^\infty} = E(\overline{K_\lambda})_{p^{m_n}} \quad \text{where } \lambda \mid \ell_n(i) \text{ and } i \in \{1, \dots, t\}.$$

Observe that if  $s \in H_{\text{Sel}}^1(K, E_{p^{m_n}})$  is an eigenvector of the complex conjugation  $\tau$  and if, viewed as an element of  $\text{Hom}_{\mathcal{G}_n}(\text{Gal}(\overline{L}_n/L_n), E_{p^{m_n}})$ , it is trivial on  $\text{Gal}(M_n/L_n)^+$ , then  $s(\text{Gal}(M_n/L_n))$  is a  $\mathcal{G}_n$ -invariant submodule of one of the eigenspaces of  $E_{p^{m_n}}$ . Since we have assumed that  $\text{Gal}(\mathbb{Q}(E_p)/\mathbb{Q})$  is not solvable, it follows that  $s(\text{Gal}(M_n/L_n)) = 0$ . Hence, by the choice of  $\{h_n(1), \dots, h_n(t)\}$ , we know that if  $s \in H_{\text{Sel}}^1(K, E_{p^{m_n}})^\pm$  and  $s(h_n(i)) = 0$  for all  $i \in \{1, \dots, t\}$ , then  $s = 0$ . By [Gro91, Proposition 9.6] and [ÇW08, Proposition 2.4.2], we have that for any  $s \in H_{\text{Sel}}^1(K, E_{p^{m_n}})$ ,

$$\text{res}_{\lambda_n(i)} s = 0 \quad \text{if and only if } s(h_n(i)) = 0.$$

Since  $H_{\text{Sel}}^1(K, E_{p^{m_n}})$  is the direct sum of its eigenspaces under the action of  $\tau$ , we can conclude that the map

$$H_{\text{Sel}}^1(K, E_{p^{m_n}}) \rightarrow \prod_{i=1}^t H^1(K_{\lambda_n(i)}, E_{p^{m_n}})$$

is injective. We have now shown that the set  $Q_n = \{\ell_n(1), \dots, \ell_n(t)\}$  satisfies the first four properties.

We shall now refine the choice of primes in  $Q_n$  to ensure that the last two properties are satisfied. Let  $h_n \in \text{Gal}(\overline{L}_n/K_n L_n)$ . In [ÇW08, § 2.5.2] we defined the  $R_n$ -module  $[R_n \alpha_n](h_n)$  as follows:

$$[R_n \alpha_n](h_n) = \left\{ \sum_{i=1}^{p^{2n}} [(g^{-i}c)(h_n)] \cdot g^i \text{ such that } c \in R_n \alpha_n \right\} \subseteq \text{Hom}_{\text{sets}}(G_n, E_{p^{m_n}}),$$

where  $G_n = \langle g \rangle$  and  $[(g^{-i}c)(h_n)] \in E_{p^{m_n}}$  is simply the evaluation of the class  $g^{-i}c$  at  $h_n \in \text{Gal}(\overline{K}_n(E_{p^{m_n}})/K_n(E_{p^{m_n}}))$ . The action of  $G_n$  on this module is the one induced by the standard action on  $\text{Hom}_{\text{sets}}(G_n, E_{p^{m_n}})$ , namely by multiplication on  $G_n$ ,  $(gf)(g_1) = f(gg_1)$ . The map

$R_n\alpha_n \rightarrow [R_n\alpha_n](h_n)$  is seen to be an  $R_n$ -module homomorphism. By picking a basis for  $E_{p^{mn}}$ , we view the right-hand side as  $R_n^2$  and hence view  $[R_n\alpha_n](h_n)$  as a submodule of  $R_n^2$ .

By [CW08, Lemma 2.5.3], we know that  $K_\infty$  and  $L_n$  are disjoint over  $K$ . Since we are assuming that  $\text{Gal}(\mathbb{Q}(E_p)/\mathbb{Q})$  is not solvable, it follows that  $M_n$  and  $K_n$  are disjoint over  $K$ . Hence we can assume that  $h_n(i) \in \text{Gal}(\overline{L_n}/K_nL_n)$ . Then, by [CW08, Proposition 2.5.7], we know that

$$\text{res}_{\ell_n(i)}(R_n\alpha_n) \simeq [R_n\alpha_n](h_n(i)) \text{ as } R_n\text{-modules.}$$

Let  $(h_n(i))_{n \in \mathbb{N}} \in \text{Gal}(\overline{L_\infty}/L_\infty)$ , where  $h_n(i) \in \text{Gal}(\overline{L_n}/K_nL_n)$  and  $i \in \{1, \dots, t\}$ . As above, we have that

$$\text{res}_{\ell_m(i)}(R_n\alpha_n) \simeq [R_n\alpha_n](h_m(i)) \quad \text{for all } m \geq n$$

and, moreover, the compatibility of  $h_n(i) \in \text{Gal}(\overline{L_n}/K_nL_n)$  implies that

$$[R_{2n}\alpha_{2n} + R_{2n+1}\alpha_{2n+1}](h_{2n+1}(i)) = [R_{2n}\alpha_{2n} + R_{2n+1}\alpha_{2n+1}](h_m(i)) \quad \text{for all } m \geq 2n + 1.$$

Hence we have

$$\begin{aligned} \text{res}_{\ell_{2n+1}(i)}(R_{2n}\alpha_{2n} + R_{2n+1}\alpha_{2n+1}) &\simeq [R_{2n}\alpha_{2n} + R_{2n+1}\alpha_{2n+1}](h_{2n+1}(i)) \\ &= [R_{2n}\alpha_{2n} + R_{2n+1}\alpha_{2n+1}](h_m(i)) \\ &\simeq \text{res}_{\ell_m(i)}(R_{2n}\alpha_{2n} + R_{2n+1}\alpha_{2n+1}) \end{aligned}$$

for all  $m \geq 2n + 1$ . This concludes the proof of part (v) of this proposition.

By the compatibility of  $h_n(i) \in \text{Gal}(\overline{L_n}/K_nL_n)$  and the fact that  $R_n\alpha_n \hookrightarrow R_{n+2}\alpha_{n+2}$  under the map

$$H^1(K_n, E_{p^{mn}}) \rightarrow H^1(K_{n+2}, E_{p^{m_{n+2}}}),$$

we have that

$$[R_n\alpha_n](h_n(i)) = [R_n\alpha_n](h_{n+2}(i)) \hookrightarrow [R_{n+2}\alpha_{n+2}](h_{n+2}(i)) \quad \text{for every } n \in \mathbb{N}.$$

By choosing the basis of  $E_{p^{mn}}$  compatibly as  $n$  grows, we can consider the direct limit  $\varinjlim_n [R_{2n}\alpha_{2n} + R_{2n+1}\alpha_{2n+1}](h_{2n+1}(i))$  and view it as a  $\Lambda$ -submodule of  $\hat{\Lambda}^2$ .

By observing that the diagram

$$\begin{array}{ccc} R_n\alpha_n & \longrightarrow & [R_n\alpha_n](h_n(i)) \\ \downarrow & & \downarrow \\ R_{n+2}\alpha_{n+2} & \longrightarrow & [R_{n+2}\alpha_{n+2}](h_{n+2}(i)) \end{array}$$

is commutative, we deduce that there is the following surjective map of  $\Lambda$ -modules:

$$\psi : \varinjlim_n (R_{2n}\alpha_{2n} + R_{2n+1}\alpha_{2n+1}) \rightarrow \varinjlim_n [R_{2n}\alpha_{2n} + R_{2n+1}\alpha_{2n+1}](h_{2n+1}(i)).$$

In [CW08, § 2.6.4], we used the first property of Kolyvagin's classes and the fact that the module  $\varinjlim_n (R_{2n}\alpha_{2n} + R_{2n+1}\alpha_{2n+1})$  has  $\Lambda$ -corank at least two (Proposition 2.1) to show that we can choose  $(h_n(i))_{n \in \mathbb{N}} \in \text{Gal}(\overline{L_\infty}/L_\infty)$  such that:

- (i)  $h_n(i) \in \text{Gal}(\overline{L_n}/K_nL_n)$  and the restriction of  $h_n(i)$  to  $M_n$  lies in  $\text{Gal}(M_n/L_n)^+$ ;
- (ii)  $\langle h_n(1), \dots, h_n(t) \rangle = \text{Gal}(M_n/L_n)^+$ ;

(iii) the invariants of  $f \varinjlim_n [\mathbb{R}_{2n}\alpha_{2n} + \mathbb{R}_{2n+1}\alpha_{2n+1}](h_{2n+1}(i))$  contain a subgroup isomorphic to  $(\mathbb{Q}_p/\mathbb{Z}_p)^2$  for all  $f \in \Lambda$ , which implies that  $\varinjlim_n [\mathbb{R}_{2n}\alpha_{2n} + \mathbb{R}_{2n+1}\alpha_{2n+1}](h_{2n+1}(i))$  has  $\Lambda$ -corank two.

By part (v), we have the following diagram.

$$\begin{array}{ccc} \text{res}_{\ell_{2n+1}(i)}(\mathbb{R}_{2n}\alpha_{2n} + \mathbb{R}_{2n+1}\alpha_{2n+1}) & \xrightarrow{\cong} & [\mathbb{R}_{2n}\alpha_{2n} + \mathbb{R}_{2n+1}\alpha_{2n+1}](h_{2n+1}(i)) \\ & & \downarrow \\ \text{res}_{\ell_{2n+3}(i)}(\mathbb{R}_{2n+2}\alpha_{2n+2} + \mathbb{R}_{2n+3}\alpha_{2n+3}) & \xrightarrow{\cong} & [\mathbb{R}_{2n+2}\alpha_{2n+2} + \mathbb{R}_{2n+3}\alpha_{2n+3}](h_{2n+3}(i)) \end{array} \quad (13)$$

This allows us to see that we can define injective maps

$$\text{res}_{\ell_{2n+1}(i)}(\mathbb{R}_{2n}\alpha_{2n} + \mathbb{R}_{2n+1}\alpha_{2n+1}) \hookrightarrow \text{res}_{\ell_{2n+3}(i)}(\mathbb{R}_{2n+2}\alpha_{2n+2} + \mathbb{R}_{2n+3}\alpha_{2n+3}) \quad (14)$$

which transform (13) into a commutative diagram. We use the above maps to construct the direct limit  $\varinjlim_n \text{res}_{\ell_{2n+1}(i)}(\mathbb{R}_{2n}\alpha_{2n} + \mathbb{R}_{2n+1}\alpha_{2n+1})$ , and then we have that

$$\varinjlim_n \text{res}_{\ell_{2n+1}(i)}(\mathbb{R}_{2n}\alpha_{2n} + \mathbb{R}_{2n+1}\alpha_{2n+1}) \simeq \varinjlim_n [\mathbb{R}_{2n}\alpha_{2n} + \mathbb{R}_{2n+1}\alpha_{2n+1}](h_{2n+1}(i)).$$

It follows that the formal direct limit  $\varinjlim_n \text{res}_{\ell_{2n+1}(i)}(\mathbb{R}_{2n}\alpha_{2n} + \mathbb{R}_{2n+1}\alpha_{2n+1})$  has  $\Lambda$ -corank two for each  $i \in \{1, \dots, t\}$ . Hence the set  $\mathbb{Q}_n$  satisfies all the required properties.  $\square$

### 3. The $\Lambda$ -corank of the Tate–Shafarevich group

We will now use Kolyvagin’s classes to analyze the image of the map

$$\mathbb{H}^1_{\text{Sel}_p \cup \mathbb{Q}_{k_{2n+1}}}(\mathbb{K}_{2n+1}, \mathbb{E}_p^{m_{2n+1}}) \rightarrow \prod_{q \in \mathbb{Q}_{k_{2n+1}}} \mathbb{H}^1(\mathbb{K}_{2n+1}(q), \mathbb{E})_p^{m_{2n+1}}, \quad (15)$$

where  $\mathbb{Q}_n$  is the set of primes chosen in Proposition 2.3. Using properties (2) and (3) of Kolyvagin’s classes, we can see that the image of

$$\begin{aligned} & \mathbb{R}_{2n}c_{2n}(\ell_{k_{2n+1}}(1)) + \mathbb{R}_{2n+1}c_{2n+1}(\ell_{k_{2n+1}}(1)) + \dots + \mathbb{R}_{2n}c_{2n}(\ell_{k_{2n+1}}(t)) \\ & + \mathbb{R}_{2n+1}c_{2n+1}(\ell_{k_{2n+1}}(t)) \subseteq \mathbb{H}^1_{\text{Sel}_p \cup \mathbb{Q}_{k_{2n+1}}}(\mathbb{K}_{2n+1}, \mathbb{E}_p^{m_{2n+1}}) \end{aligned}$$

under the map (15) is

$$\prod_{i=1}^t \text{res}_{\ell_{k_{2n+1}}(i)}[\mathbb{R}_{2n}d_{2n}(\ell_{k_{2n+1}}(i)) + \mathbb{R}_{2n+1}d_{2n+1}(\ell_{k_{2n+1}}(i))].$$

We know that the maps  $\psi_{\ell_{k_{2n+1}}(i)}$ , from property (3) of Kolyvagin’s classes, induce the isomorphisms

$$\text{res}_{\ell_{k_{2n+1}}(i)}[\mathbb{R}_{2n}d_{2n}(\ell_{k_{2n+1}}(i)) + \mathbb{R}_{2n+1}d_{2n+1}(\ell_{k_{2n+1}}(i))] \simeq \text{res}_{\ell_{k_{2n+1}}(i)}[\mathbb{R}_{2n}\alpha_{2n} + \mathbb{R}_{2n+1}\alpha_{2n+1}]$$

for each  $i = 1, \dots, t$ . We now use the maps (14) to define the injective maps

$$\begin{aligned} & \text{res}_{\ell_{k_{2n+1}}(i)}[\mathbf{R}_{2n}d_{2n}(\ell_{k_{2n+1}}(i)) + \mathbf{R}_{2n+1}d_{2n+1}(\ell_{k_{2n+1}}(i))] \\ & \hookrightarrow \text{res}_{\ell_{k_{2n+3}}(i)}[\mathbf{R}_{2n+2}d_{2n+2}(\ell_{k_{2n+3}}(i)) + \mathbf{R}_{2n+3}d_{2n+3}(\ell_{k_{2n+3}}(i))], \end{aligned}$$

which can be used as transition maps in defining the direct limit

$$\varinjlim_n \text{res}_{\ell_{k_{2n+1}}(i)}[\mathbf{R}_{2n}d_{2n}(\ell_{k_{2n+1}}(i)) + \mathbf{R}_{2n+1}d_{2n+1}(\ell_{k_{2n+1}}(i))].$$

We can immediately see that

$$\begin{aligned} & \varinjlim_n \text{res}_{\ell_{k_{2n+1}}(i)}[\mathbf{R}_{2n}d_{2n}(\ell_{k_{2n+1}}(i)) + \mathbf{R}_{2n+1}d_{2n+1}(\ell_{k_{2n+1}}(i))] \\ & \simeq \varinjlim_n \text{res}_{\ell_{2n+1}(i)}(\mathbf{R}_{2n}\alpha_{2n} + \mathbf{R}_{2n+1}\alpha_{2n+1}). \end{aligned}$$

Since, by Proposition 2.3(v), the  $\Lambda$ -modules  $\varinjlim_n \text{res}_{\ell_{2n+1}(i)}(\mathbf{R}_{2n}\alpha_{2n} + \mathbf{R}_{2n+1}\alpha_{2n+1})$  have corank two, it follows that the formal direct limit

$$\varinjlim_n \text{res}_{\ell_{k_{2n+1}}(i)}[\mathbf{R}_{2n}d_{2n}(\ell_{k_{2n+1}}(i)) + \mathbf{R}_{2n+1}d_{2n+1}(\ell_{k_{2n+1}}(i))]$$

has  $\Lambda$ -corank two for each  $i \in \{1, \dots, t\}$ . The fact that all the transition maps that we are using are injective implies that the image of the formal map  $\theta$  (see §1) has corank  $2t$ , even if the modules  $\varinjlim_n \text{res}_{\ell_{k_{2n+1}}(i)}(\mathbf{R}_{2n}\alpha_{2n} + \mathbf{R}_{2n+1}\alpha_{2n+1})$  cannot be viewed as submodules of the image of  $\theta$ . It then follows that the kernel of  $\theta$  has  $\Lambda$ -corank two. Proposition 1.3 implies that we have now proven the following theorem.

**THEOREM 3.1.** *The  $\Lambda$ -module  $H_{\text{Sel}_p}^1(\mathbf{K}_\infty, E_{p^\infty})$  has corank two.*

By Proposition 2.1, we know that the image of  $E(\mathbf{K}_\infty)$  in  $H_{\text{Sel}}^1(\mathbf{K}_\infty, E_{p^\infty})$  has  $\Lambda$ -corank at least two. Hence Theorem 3.1 implies this corollary.

**COROLLARY 3.2.** *The  $\Lambda$ -module  $E(\mathbf{K}_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p$  has corank two.*

Then, the exactness of the sequence

$$0 \rightarrow E(\mathbf{K}_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow H_{\text{Sel}}^1(\mathbf{K}_\infty, E_{p^\infty}) \rightarrow \text{III}(\mathbf{K}_\infty, E)_{p^\infty} \rightarrow 0$$

implies that the  $\Lambda$ -corank of  $\text{III}(\mathbf{K}_\infty, E)_{p^\infty}$  is trivial. This concludes the proof of Theorem 0.1.

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