ONE-WAY ANALYSIS OF VARIANCE MODEL

The simplest type of analysis of variance considers a situation that generalizes the two-sample t-test situation by allowing more than two groups.

The situation:

1. We have a response variable $Y$.

2. We have $v$ populations $G_1, G_2, \ldots, G_v$ on which the response variable is defined. In an experiment, these will be defined by the levels of a treatment factor ("treatment groups"): $G_i$ is the population that has received the $i^{th}$ treatment (that is, the $i^{th}$ level of the treatment factor).

3. Let $Y_i$ denote the response for the population $G_i$. (i.e., $Y_i = Y|G_i$, $Y$ restricted to $G_i$.)

4. Let $\mu_i$ be the mean of $Y$ on the $i^{th}$ population $G_i$. (i.e., $\mu_i = E(Y_i) = E(Y|G_i)$ ($\mu_i$ is sometimes called the true mean for the $i^{th}$ treatment or population.)

5. Let $\epsilon_i = Y_i - \mu_i$. ($\epsilon_i$ is a new random variable, called the $i^{th}$ error)

Example: Continuing with the example of testing computer packages to teach a programming language, suppose we were comparing 3 such packages rather than 2. Then $Y =$ score on the test given to the subjects $v = 3$ $G_i =$ all conceivable engineers who have used the $i^{th}$ package. $\mu_i =$ mean score of all conceivable engineers who have used the $i^{th}$ package.

Note that (5) can be re-expressed as

$Y_i = \mu_i + \epsilon_i$

(These model equation is sometimes called a linear or additive model. This form of the model equations is called the means model.)

Model assumptions:

1. For each $i$, a simple random sample of size $r_i$ from population $G_i$ is gathered.
2. The samples are independent.

3. Each error variable $\varepsilon_i$ is normally distributed.

4. All error variables $\varepsilon_i$ have the same variance $\sigma^2$.

Comments:
1. For an experiment, assumptions (1) and (2) can be combined to say that experimental units are randomly assigned to treatments, subject only to the constraint that the sample size for the $i^{th}$ treatment is $r_i$. An experimental design in which experimental units are randomly assigned to treatments (subject only to the constraint that the sample size for the $i^{th}$ treatment is $r_i$) is said to be completely randomized.

2. If all the $r_i$'s are equal, then the design is said to be balanced.

3. Assumptions (3) and (4) can be combined to say $\varepsilon_i \sim N(\mu_i, \sigma^2)$

4. Students who have had regression should recognize the similarities to a linear regression model with indicator variables representing a categorical variable.

Alternate formulations of the model equation.

1. Let $\mu = E(Y)$ (the overall mean) and let $\tau_i = \mu_i - \mu$. Then the model equation can be rephrased as

$$Y_i = \mu + \tau_i + \varepsilon_i.$$  

($\tau_i$ is called the effect of the $i^{th}$ treatment on the response. This version of the model is sometimes called the effects model.)

2. Many people prefer to state the model equation in terms of the sample random variables as follows:

Let $Y_{it}$ be the random variable that represents the response from the $t^{th}$ observation from $G_i$ (in terms of an experiment: the response from the $t^{th}$ observation of the $i^{th}$ treatment). Let $\varepsilon_{it} = Y_{it} - \mu_i$. Then the model equation is stated as:

$$Y_{it} = \mu_i + \varepsilon_{it}$$

or $$Y_{it} = \mu + \tau_i + \varepsilon_{it}.$$
In this formulation, the model assumptions become:
   a) The $\varepsilon_{it}$ are independent random variables.
   b) For each i and t, $\varepsilon_{it} \sim N(\mu, \sigma^2)$

Note: The alternate model formulation as presented here is called a fixed effects model, since we are assuming that we have specified treatments fixed by the experimenter. Note that in the fixed effects model, the $\tau_i$'s are parameters. This model could be generalized to a situation where the treatments are a random sample from a larger population of treatments. In this case, the $\tau_i$'s are random variables. This generalization is called a random effects model. Random effects models are discussed in Chapter 17.

Dot notation

The following notational conventions are convenient when we are dealing with lots of subscripts:

- A dot in a subscript position means "add over all values of the subscript in that position."
- If there is a bar over the variable as well as a dot in the subscript position, then divide by the number of possibilities for the subscript as well as add over all values of the subscript. (In other words, take the average over all values of the subscript.)

Examples:

$$Y_i = \sum_{t=1}^{\nu} Y_{it} \quad \bar{Y}_{.i} = \frac{1}{\nu} \sum_{t=1}^{\nu} Y_{it} \quad Y_{.i} = \sum_{i=1}^{\nu} Y_{it} \quad Y = \sum_{i=1}^{\nu} \sum_{t=1}^{\nu} Y_{it}$$