INFERENCE FOR ONE-WAY ANOVA

To test equality of means for different treatments/levels, we can use the null hypothesis

\[ H_0: \mu_1 = \mu_2 = \ldots = \mu_v \]

Rephrase:
1. In terms of effects: _________________________________
2. In terms of differences of effects: _________________________________
3. In terms of contrasts \( \tau_i - \bar{\tau} \), where \( \bar{\tau} = \frac{1}{v} \sum_{i=1}^{v} \tau_i \): _________________________________

The *treatment degrees of freedom* is the minimum number of equations needed to state the null hypothesis, in other words __________________.

Alternate hypothesis: \( H_a: _______________________________ \)

Idea of the test: Compare ssE under the *full* model (with all parameters) with the error sum of squares ssE₀ under the *reduced* model -- i.e., the one assuming \( H_0 \) is true.

To calculate ssE₀: If \( H_0 \) is true, let \( \tau \) be the common value of the \( \tau_i \)'s. Then the reduced model is

- \( Y_{it} = \mu + \tau + \varepsilon_{it}^0 \)
- \( \varepsilon_{it}^0 \sim N(0, \sigma^2) \)
- the \( \varepsilon_{it}^0 \)'s are independent,

where \( \varepsilon_{it}^0 \) denotes the \( it^{th} \) error in the reduced model.

To find ssE₀, we use least squares to minimize \( g(m) = \sum_{i=1}^{v} \sum_{t=1}^{r_i} (y_{it} - m)^2 : \)

\[ g'(m) = \sum_{i=1}^{v} \sum_{t=1}^{r_i} 2(-1)(y_{it} - m) = 0, \]
which yields estimate \( \bar{y}.. \) for \( \mu + \tau \) -- that is, the least squares estimate of \( \mu + \tau \) is \( (\mu + \tau)^\wedge = \bar{y}.. \). (By abuse of notation, we call this \( \hat{\mu} + \hat{\tau} \)). So

\[ \text{ssE}_0 = \sum_{i=1}^{v} \sum_{t=1}^{r_i} (y_{it} - \bar{y}..)^2, \]
which can be shown (proof might be homework) to equal \( \sum_{i=1}^{v} \sum_{t=1}^{r_i} y_{it}^2 - n(\bar{y}..)^2 \)
Note that ssE and ssE₀ can be considered as minimizing the same expression, but over different sets: ssE minimizes \( \sum_{i=1}^{v} \sum_{t=1}^{r_i} (y_{it} - m - t^i)^2 \) over the set of all \( v + 1 \)-tuples \((m, t_1, t_2, \ldots, t_v)\), whereas ssE₀ can be considered as minimizing the same expression over the subset where all \( t_i \)'s are zero. Thus ssE₀ must be at least as large as ssE: \( ssE_0 \geq ssE \).

However, if \( H_0 \) is true, then ssE and ssE₀ should be about the same. This suggests the idea of using the ratio \((ssE_0 - ssE)/ssE\) as a test for the null hypothesis: If \( H_0 \) is true, this ratio should be small; so an unusually large ratio would be reason to reject the null hypothesis.

The difference \( ssE_0 - ssE \) is called the *sum of squares for treatment, or treatment sum of squares*, denoted ssT. Using the alternate expressions for ssE₀ and ssE, we have:

\[
ssT = ssE_0 - ssE = \sum_{i=1}^{v} \sum_{t=1}^{r_i} y_{it}^2 - n(y..)^2 \left( \sum_{i=1}^{v} \sum_{t=1}^{r_i} y_{it}^2 - \sum_{i=1}^{v} r_i (\bar{y}_i)^2 \right) \\
= \sum_{i=1}^{v} r_i (\bar{y}_i)^2 - n(\bar{y}..)^2 \\
= \sum_{i=1}^{v} \frac{(y_i^2)}{r_i} - \frac{(y..)^2}{n} \quad \text{(using definitions)} \\
= \sum_{i=1}^{v} r_i (\bar{y}_i - \bar{y}..)^2 \quad \text{(possible homework)}
\]

This last expression can be considered as a "between treatments" sum of squares --- we are comparing each treatment sample mean \( \bar{y}_i \) with the grand (overall) mean \( \bar{y}.. \). By contrast, our denominator, ssE = \( \sum_{i=1}^{v} \sum_{t=1}^{r_i} (y_{it} - \bar{y}_i)^2 \) is a "within treatments" sum of squares: it compares each value with the mean for the treatment group from which the value was obtained.

Using the model assumptions, it can be proved that:
- \( ssE/\sigma^2 \sim \chi^2(n - v) \)
- If \( H_0 \) is true, \( ssT/\sigma^2 \sim \chi^2(v - 1) \)
- If \( H_0 \) is true, then ssT and ssE are independent.

Thus, if \( H_0 \) is true,

\[
\frac{ssT/\sigma^2(v - 1)}{ssE/\sigma^2(n - v)} \sim F_{v - 1, n - v}.
\]
Now \( \frac{ssT}{\sigma^2(v - 1)} \) simplifies to \( \frac{ssT(v - 1)}{ssE(n - v)} \), which we can calculate from our sample.

We originally wanted to test \( ssT/ssE \), but \( \frac{ssT(v - 1)}{ssE(n - v)} \) is just a constant multiple of \( ssT/ssE \), so is good enough for our purposes: \( \frac{ssT(v - 1)}{ssE(n - v)} \) will be unusually large exactly when \( ssT/ssE \) is unusually large. Thus, we can use an F test, with test statistic \( \frac{ssT(v - 1)}{ssE(n - v)} \), to test our hypothesis.

**Note:** We can look at \( ssT/(v-1) \) and \( ssE/(n-v) \) as we did in the equal-variance, two-sample t-test: \( ssE/(n-v) \) is a pooled estimate of the common variance \( \sigma^2 \), and if \( H_0 \) is true, then \( ssT/(v - 1) \) can be regarded as an estimate of \( \sigma^2 \).

**Notation:** \( ssT/(v-1) \) is called msT (mean square for treatment or treatment mean square) and \( ssE/(n-v) \) is called msE (mean square for error or error mean square). So the test statistic is \( F = msT/msE \).