

Notes for M 384E, Wednesday, January 21, 2009

(*Please note:* I will *not* pass out hard-copy class notes in future classes. If there are written class notes, they will be posted on the web by the night before class for you to download yourself.)

This is an introductory course in Analysis of Variance and Design of Experiments.

## I. BRIEF OVERVIEW

*Questions:*

1. What is Analysis of Variance?
2. How is Analysis of Variance connected to Design of Experiments?

*Brief Answers:*

1. Analysis of Variance (ANOVA) is a methodology that can be used for statistical inference in a variety of situations generalizing the equal-variance two-sample t-test.
2. The details of implementation of ANOVA depend on the design of the method for collecting data -- typically, by an experiment. The design needs to take into account the methods of analysis as well as the particulars of the context (the question of interest, factors that may influence the variable of interest, and constraints such as time and budget.)

## II. MODEL ASSUMPTIONS FOR STATISTICAL PROCEDURES

Statistical procedures typically have certain *model assumptions*. These are assumptions about the distributions of random variables involved, or about how samples are chosen, or about the type of relationship between the variables involved. The essence of applying statistics is to find a model that does all of the following:

1. Fits the real-world situation involved well enough.
2. Leads to a valid method of statistical analysis.
3. Gives information relevant to the questions of interest.

In using statistics, always bear in mind the words of the statistician G.E. Box:

***All models are wrong; some are useful.***

This means that models never fit the real-world situation exactly, but we need to be sure that they fit "well enough" and that they give relevant information.

## III. REVIEW OF THE EQUAL VARIANCE TWO-SAMPLE T-TEST (focusing on the model assumptions and why they are important.)

(*Note*: There is another two-sample t-test that does not assume equal variance. However, the equal variance test is the one we will be concerned with here, since it is the one that generalizes to the Analysis of Variance method.)

*Be sure to review the handout Review of Basic Statistical Concepts if you are confused about notation or basic concepts used below.*

In discussing the equal variance two-sample t-test, we will focus on the model assumptions and why they are important for good application of the method.

**Model Assumptions for the equal variance two-sample t- test:**

1.  $x_1, x_2, \dots, x_m$  and  $y_1, y_2, \dots, y_n$  are *independent, random* samples from random variables X and Y.
2. X and Y are each *normally distributed*.
3. X and Y have the *same variance* (which is not known)

Denote the means of X and Y by  $\mu_X$  and  $\mu_Y$ , respectively. (These are *population means*, not to be confused with sample means  $\bar{x}$  and  $\bar{y}$ )

We wish to test the null hypothesis  $H_0: \mu_X = \mu_Y$   
against the two-sided alternative  $H_a: \mu_X \neq \mu_Y$

**Example:** A large company is planning to purchase a large quantity of computer packages designed to teach a new programming language. A consultant claims that the two packages are equal in effectiveness. To test this claim, the company randomly selects 60 engineers and randomly assigns 30 to use the first package and 30 to use the second package. Each engineer is given a standardized test of programming skill after completing the training with the assigned package. The scores of the 30 engineers assigned to the first package are  $x_1, x_2, \dots, x_m$ ; the scores of those assigned to the second package are  $y_1, y_2, \dots, y_n$ . (In this example,  $n = m = 30$ .) The random variable X is "test score of an engineer from this company using the first package." The random variable Y is "test score of an engineer from this company using the second package". Since the engineers are randomly chosen and randomly assigned to the package, assumption (1) is satisfied. Since the test, like most standardized tests, is devised and scored to have a normal distribution of scores, assumption (2) is reasonable. It is plausible (we hope) to assume that the variability in scores will not depend on the package chosen, so assumption (3) seems reasonable (although perhaps we might want to look the data to get an additional check on whether this assumption is reasonable).

**Outline of what the test involves and why it works** (focusing on where the model assumptions are needed):

(For more details, see Ross, Section 4.2 or Wackerly Section 10.8)

Denote the (unknown) variance of X and Y by  $\sigma^2$ .

*Notation:* The notation  $X \sim N(\mu_X, \sigma^2)$  is short for "the random variable  $X$  is normally distributed with mean  $\mu_X$  and variance  $\sigma^2$ ". Thus from our assumptions:

$$X \sim N(\mu_X, \sigma^2) \text{ and } Y \sim N(\mu_Y, \sigma^2)$$

From our sample  $y_1, y_2, \dots, y_n$ , we can calculate the sample mean  $\bar{y}$ , which is our best estimate of the mean  $\mu_Y$ . We could also calculate the sample mean for *any* random sample of size  $n$  chosen from  $Y$ . This process ("take a random sample of size  $n$  from  $Y$  and calculate its sample mean") describes a new random variable, which we will call  $\bar{Y}$ . Thus,  $\bar{y}$  is the value of the random variable  $\bar{Y}$  obtained by picking our particular sample. Since  $\bar{Y}$  is a random variable, it has a distribution (called a *sampling distribution*, since the value of  $\bar{Y}$  depends on the sample chosen). Mathematical theory tells us that the random variable  $\bar{Y}$  is normally distributed with mean  $\mu_Y$  and variance  $\sigma^2/n$ :  
 $\bar{Y} \sim N(\mu_Y, \sigma^2/n)$

*This conclusion uses the following facts (or assumptions, as the case may be):*

- $y_1, y_2, \dots, y_n$  is a random sample
- $Y \sim N(\mu_Y, \sigma^2)$ .

Similarly (given the model assumptions),  $\bar{X} \sim N(\mu_X, \sigma^2/m)$

Our hypotheses can be restated in terms of the difference  $\mu_X - \mu_Y$ :

$$H_0: \mu_X - \mu_Y = 0 \quad H_a: \mu_X - \mu_Y \neq 0$$

Thus we consider the difference  $\bar{x} - \bar{y}$  as an *estimate* of  $\mu_X - \mu_Y$ . In the language of random variables,  $\bar{X} - \bar{Y}$  is an *estimator* of  $\mu_X - \mu_Y$ .

Since our samples from  $X$  and  $Y$  are *independent*, the random variables  $\bar{X}$  and  $\bar{Y}$  are also independent. Mathematical theory tells us that:

1. The sum of *independent normal* random variables is normal (so we know that  $\bar{X} - \bar{Y}$  is normal).
2. The mean (expected value) of the sum of random variables is the sum of the means of the terms (so we know that the mean of  $\bar{X} - \bar{Y}$  is  $\mu_X - \mu_Y$ ).
3. The variance of the sum or difference of *independent* random variables is the sum of the variances of the terms (so we know that  $\text{Var}(\bar{X} - \bar{Y}) = \text{Var}(\bar{X}) + \text{Var}(\bar{Y}) = \sigma^2/m + \sigma^2/n$ ).

Thus we have:

$$\bar{X} - \bar{Y} \sim N(\mu_X - \mu_Y, \sigma^2/m + \sigma^2/n).$$

Therefore

$$\frac{\bar{X} - \bar{Y} - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma^2}{m} + \frac{\sigma^2}{n}}} \sim N(0,1) \quad (\text{i.e., is standard normal}).$$

If we knew  $\sigma^2$ , this would give us a test statistic to do inference on  $\mu_X - \mu_Y$ . But we don't know  $\sigma^2$ . We do, however, have two estimates of  $\sigma^2$ : the two *sample variances*

$$s_X^2 = \sum_{i=1}^m \frac{(x_i - \bar{x})^2}{m-1} \quad \text{and} \quad s_Y^2 = \sum_{i=1}^n \frac{(y_i - \bar{y})^2}{n-1}$$

Consistently with our notation  $\bar{y}$  and  $\bar{Y}$  above, we will use capital letters to refer to the underlying random variables (the *estimators* of  $\sigma^2$ ):

$$S_X^2 = \sum_{i=1}^m \frac{(X_i - \bar{X})^2}{m-1} \quad \text{and} \quad S_Y^2 = \sum_{i=1}^n \frac{(Y_i - \bar{Y})^2}{n-1}$$

Which of these two estimators should we use? What seems better than picking one or the other is taking their average. But since they are come from possibly different sized samples, we use their *weighted* mean, yielding the *pooled estimator*

$$S^2 = \frac{(m-1)S_X^2 + (n-1)S_Y^2}{(m-1) + (n-1)} = \frac{1}{m+n-2} \left[ \sum_{i=1}^m (x_i - \bar{x})^2 + \sum_{i=1}^n (y_i - \bar{y})^2 \right].$$

So we consider the random variable  $T = \frac{\bar{X} - \bar{Y} - (\mu_X - \mu_Y)}{\sqrt{\frac{S^2}{m} + \frac{S^2}{n}}}$

Mathematical theory (using the model assumptions) tells us that T has a t-distribution with  $n + m - 2$  degrees of freedom. If  $H_0$  is true, then T is just

$$\frac{\bar{X} - \bar{Y}}{\sqrt{\frac{S^2}{m} + \frac{S^2}{n}}} = \frac{\bar{X} - \bar{Y}}{S \sqrt{\frac{1}{m} + \frac{1}{n}}}.$$

Summarizing: If  $H_0$  is true, then the random value  $T = \frac{\bar{X} - \bar{Y}}{S \sqrt{\frac{1}{m} + \frac{1}{n}}}$  has a t-distribution

with  $n + m - 2$  degrees of freedom. We can use this fact to perform our hypothesis test: Calculate the value t of T determined by our sample. Calculate the corresponding p-value:

$p$  = the probability of obtaining a value of T (having a t-distribution with  $n + m - 2$  degrees of freedom) with absolute value greater than or equal to  $|t|$ .

If  $p$  is sufficiently small, we choose to reject the null hypothesis in favor of the alternate.  
 → Note that this is *not* the same as saying  $H_a$  is true, and is also *not* the same as saying

that  $H_0$  is false; it just says that  $H_a$  appears to be the better option, given the evidence at hand.

*Otherwise*, we do not reject  $H_0$  -- the evidence is consistent with it.

→ Note that this is *not* the same as saying that  $H_0$  is true, and it is also *not* the same as saying  $H_a$  is false; it's just saying that there is no reason to prefer  $H_a$  to  $H_0$ , given the evidence at hand.

**Example:** Continuing with the example of comparing the two packages for teaching a new programming language, if we obtain sample mean 72.5 and sample standard deviation 10.3 for the first method, and sample mean and standard deviation 70.1 and 11.8, respectively, for the second method, then the *pooled sample variance* is

$$s^2 = [29(10.3^2) + 29(11.8^2)]/58 = 122.665,$$

so the *pooled standard deviation* is

$$s = \sqrt{122.665} = 11.075,$$

the *pooled standard error* (which is our estimate of the standard error of the random variable  $\bar{X} - \bar{Y}$ ) is

$$se(\bar{x} - \bar{y}) = s \sqrt{\frac{1}{30} + \frac{1}{30}} = 2.86$$

and the t-statistic is

$$\frac{72.5 - 70.1}{11.075 \sqrt{\frac{1}{30} + \frac{1}{30}}} = 2.4/2.86 = .8392$$

The corresponding p-value (two-tailed, using a t-distribution with 58 degrees of freedom) is 0.404825. This does not give us any evidence against the null hypothesis, so we have not detected any significant difference between the two packages -- we have no reason, based on the test scores, to choose one over the other.

We could also use the t-statistic to calculate a confidence interval for the difference  $\mu_x - \mu_y$  in the sample means. Suppose we want a 90% confidence interval. For a t-distribution with 58 degrees of freedom, 90% of all values lie between - 1.67155 and + 1.67155. So for 90% of all samples satisfying the model assumptions,

$$- 1.67155 < T < 1.67155.$$

In other words,

$$- 1.67155 < \frac{(\bar{X} - \bar{Y}) - (\mu_x - \mu_y)}{se(\bar{X} - \bar{Y})} < 1.67155.$$

A bit of algebra manipulation shows that this is equivalent to

$$(\bar{X} - \bar{Y}) - 1.67155\text{se}(\bar{X} - \bar{Y}) < \mu_X - \mu_Y < (\bar{X} - \bar{Y}) + 1.67155\text{se}(\bar{X} - \bar{Y}).$$

Evaluating this for our sample gives the endpoints

$$(72.5 - 70.1) \pm 1.67155(2.86)$$

for the confidence interval, resulting in confidence interval ( -2.38, 7.18).

Note that we are *not* asserting that  $\mu_X - \mu_Y$  lies in this interval. All we have done is use a procedure that, for 90% of all pairs of simple random samples of sizes  $n$ , chosen independently from the populations in question, will give an interval that does contain  $\mu_X - \mu_Y$ . Our sample could be one of the 10% yielding a confidence interval that does not contain  $\mu_X - \mu_Y$ .

Note also that the confidence interval contains zero. Thus our data are consistent with the possibility that  $\mu_X - \mu_Y = 0$  -- in other words, that  $\mu_X = \mu_Y$ . (Note that this is the same conclusion we drew from the hypothesis test. )

**References:**

Ross, Sheldon M., Introduction to Probability and Statistics for Engineers and Scientists, Wiley, 1987

Wackerly, Dennis, William Mendenhall and Richard Scheafer, Mathematical Statistics with Applications, Duxbury, 1996