

CONFIDENCE INTERVALS FOR VARIANCE COMPONENTS (Section 17.3.5)

In practice, these play the role for random effects that confidence intervals for contrasts play for fixed effects.

Confidence intervals for σ^2 : These are constructed just as for fixed effects; see Section 3.4.6 or the class notes *Choosing Sample Sizes*.

Confidence intervals involving σ_T^2 : Three types of confidence intervals are of interest: for σ_T^2 , for σ_T^2/σ^2 , and for $\sigma_T^2/(\sigma_T^2 + \sigma^2)$. The first cannot be done exactly, so we'll take that last.

Confidence intervals for σ_T^2/σ^2 : We use the fact (see notes *Testing for Treatment Effect as a Proportion of Error Variance*) that

$$\frac{MST / (c\sigma_T^2 + \sigma^2)}{MSE / (\sigma^2)} \sim F(v-1, n-v),$$

where c is a certain constant defined in terms of n, v , and the r_i ; $c = r$ if the design is balanced) (See notes *Random Effects Models* or Section 17.3)

If we want a $(1-\alpha)100\%$ CI for σ_T^2/σ^2 , take

$$f_1 = F(v-1, n-v, 1-\alpha/2) \text{ (so that there is area } \alpha/2 \text{ to the left of } f_1 \text{ in the } F(v-1, n-v) \text{ distribution), and}$$

$$f_2 = F(v-1, n-v, \alpha/2) \text{ (so that there is area } \alpha/2 \text{ to the right of } f_2 \text{ in the } F(v-1, n-v) \text{ distribution). [Draw a picture!]}$$

Then

$$\text{Prob} \left(f_1 \leq \frac{MST / (c\sigma_T^2 + \sigma^2)}{MSE / (\sigma^2)} \leq f_2 \right) = 1 - \alpha,$$

or equivalently,

$$\text{Prob} \left(f_1 \leq [MST/MSE] [\sigma^2/(c\sigma_T^2 + \sigma^2)] \leq f_2 \right) = 1 - \alpha,$$

The left inequality is equivalent to

$$(c\sigma_T^2 + \sigma^2)/\sigma^2 \leq (MST/MSE)(1/f_1), \text{ or}$$

$$c(\sigma_T^2/\sigma^2) + 1 \leq (MST/MSE)(1/f_1),$$

which is equivalent to

$$c(\sigma_T^2/\sigma^2) \leq (MST/MSE)(1/f_1) - 1$$

The right inequality is equivalent to

$(\text{MST}/\text{MSE})(1/f_2) \leq (c\sigma_T^2 + \sigma^2)/\sigma^2 = c(\sigma_T^2/\sigma^2) + 1$,
which is equivalent to

$$(\text{MST}/\text{MSE})(1/f_2) - 1 \leq c(\sigma_T^2/\sigma^2)$$

So

$$\text{Prob} ((1/c)[(\text{MST}/\text{MSE})(1/f_2) - 1] \leq \sigma_T^2/\sigma^2 \leq (1/c)[(\text{MST}/\text{MSE})(1/f_1) - 1]) = 1 - \alpha.$$

Thus $((1/c)[(\text{MST}/\text{MSE})(1/f_2) - 1] , (1/c)[(\text{MST}/\text{MSE})(1/f_1) - 1])$ is the desired confidence interval. (This means _____)

Note: Conceivably the left hand endpoint could be less than 0, which is unrealistic, If it is < 0 , do *not* give in to the temptation to replace it by zero; that would give the false impression of a smaller confidence interval than warranted.

Example: Use the loom data to find a 95% confidence interval for σ_T^2/σ^2 .

Confidence intervals for $\sigma_T^2/(\sigma_T^2 + \sigma^2)$ = the proportion of the total variance if the response attributable to the treatment level: Such confidence intervals are readily obtained from confidence intervals for σ_T^2/σ^2 as follows. Divide both numerator and denominator of $\sigma_T^2/(\sigma_T^2 + \sigma^2)$ by σ^2 to obtain

$$\sigma_T^2/(\sigma_T^2 + \sigma^2) = \frac{\sigma_T^2/\sigma^2}{\left(\frac{\sigma_T^2}{\sigma^2}\right) + 1} = f\left(\frac{\sigma_T^2}{\sigma^2}\right), \text{ where } f(x) = x/(x + 1) = \frac{1}{1 + \frac{1}{x}}.$$

From the last formula for $f(x)$, we can see that $f(x)$ is an increasing function of x . Thus if (a,b) is a $(1-\alpha)100\%$ confidence interval for σ_T^2/σ^2 , then $(f(a), f(b)) = (a/(a + 1), b/(b + 1))$ is a $(1-\alpha)100\%$ confidence interval for $\sigma_T^2/(\sigma_T^2 + \sigma^2)$.

Note: $\sigma_T^2/(\sigma_T^2 + \sigma^2)$ is sometimes called the “population intraclass correlation coefficient” (*Caution:* The phrase “intraclass correlation coefficient” is also used to refer to other things.)

Example: With the loom data, find a 95% confidence interval for $\sigma_T^2/(\sigma_T^2 + \sigma^2)$.

Confidence intervals for σ_T^2 : There is no exact method. There are several approximate methods. Here is one. It is useful if σ_T^2 is not too small, and is adaptable to more complicated models.

Recall that $U = (1/c)(\text{MST} - \text{MSE})$ is an unbiased estimator of σ_T^2 . If we knew its distribution, we could use that to get confidence intervals for σ_T^2 in the usual way.

However, U does *not* have a tractable distribution. But it *is* true that

$$U/\sigma_T^2 \approx \chi^2(x)/x, \text{ where}$$

$$x \approx \frac{(msT - msE)^2}{\frac{(msT)^2}{(v-1)} + \frac{(msE)^2}{(n-v)}}$$

(Note: This formula is given correctly on p. 605 of the text, but incorrectly on p. 600.)
 x is not usually an integer, so we need to interpret degrees of freedom in the χ^2 distribution as the parameter in a formula for the pdf. (This is analogous to the two-sample, unequal variance t-test.)

Thus (Draw a picture!)

$$P(\chi^2(x, 1 - \alpha/2) < xU/\sigma_T^2 < \chi^2(x, \alpha/2)) \approx 1 - \alpha,$$

where $<$ means “is less than or approximately equal to”, and $\chi^2(x, \beta)$ is the value with proportion β of the $\chi^2(x)$ distribution to its *right*.

The left and right approximate inequalities are, respectively, equivalent to

$$\sigma_T^2 < xU/\chi^2(x, 1 - \alpha/2) \quad \text{and} \quad \sigma_T^2 > xU/\chi^2(x, \alpha/2).$$

Thus if

$$u = (1/c)(msT - msE) \text{ (which is our estimate for } \sigma_T^2 \text{), then}$$

$(xu/\chi^2(x, \alpha/2), xu/\chi^2(x, 1 - \alpha/2))$ is an approximate $(1-\alpha)100\%$ confidence interval for σ_T^2 .

Example: With the loom data, find a 95% confidence interval for σ_T^2 .