# INFERENCE FOR ONE-WAY ANOVA

To test equality of means for different treatments/levels, we can use the null hypothesis

$H_0$: $\mu_1 = \mu_2 = \ldots = \mu_v$

Rephrase:

1. In terms of effects: ________________________________

2. In terms of differences of effects: __________________________________

3. In terms of contrasts $\tau_i - \bar{\tau}$, where $\bar{\tau} = \frac{1}{v}\sum_{i=1}^{v}\tau_i$: ________________________

The *treatment degrees of freedom* is the minimum number of equations needed to state the null hypothesis, in other words ______________.

Alternate hypothesis: $H_a$: _____________________________

Idea of the test: Compare ssE under the *full* model (with all parameters) with the error sum of squares $ssE_0$ under the *reduced* model -- i.e., the one assuming $H_0$ is true.

To calculate $ssE_0$: If $H_0$ is true, let $\tau$ be the common value of the $\tau_i$'s. Then the reduced model is

- $Y_{it} = \mu + \tau + \varepsilon_{it}^0$
- $\varepsilon_{it}^0 \sim N(0, \sigma^2)$
- the $\varepsilon_{it}^0$'s are independent,

where $\varepsilon_{it}^0$ denotes the it[th] error in the reduced model.

To find $ssE_0$, we use least squares to minimize $g(m) = \sum_{i=1}^{v}\sum_{t=1}^{r_i}(y_{it} - m)^2$:

$$g'(m) = \sum_{i=1}^{v}\sum_{t=1}^{r_i}2(-1)(y_{it} - m) = 0,$$

which yields estimate $\bar{y}_{..}$ for $\mu + \tau$ -- that is, the least squares estimate of $\mu + \tau$ is $(\mu + \tau)^\wedge = \bar{y}_{..}$ . (By abuse of notation, we call this $\hat{\mu} + \hat{\tau}$). So

$$ssE_0 = \sum_{i=1}^{v}\sum_{t=1}^{r_i}(y_{it} - \bar{y}_{..})^2,$$

which can be shown (proof might be homework) to equal $\sum_{i=1}^{v}\sum_{t=1}^{r_i} y_{it}^2 - n(\bar{y}_{..})^2$

Note that ssE and $ssE_0$ can be considered as minimizing the same expression, but over different sets: ssE minimizes $\sum_{i=1}^{t}\sum_{t=1}^{r_i}\left(y_{it}-m-t_i\right)^2$ over the set of all v + 1-tuples ($m, t_1, t_2, \ldots, t_v$), whereas $ssE_0$ can be considered as minimizing the same expression over the subset where all $t_i$'s are zero. Thus $ssE_0$ must be at least as large as ssE: $ssE_0 \geq ssE$.

However, if $H_0$ is true, then ssE and $ssE_0$ should be about the same. This suggests the idea of using the ratio ($ssE_0$-ssE)/ssE as a test statistic for the null hypothesis: If $H_0$ is true, this
ratio should be small; so an ususually large ratio would be reason to reject the null hypothesis.

The difference $ssE_0$-ssE is called the *sum of squares for treatment,* or *treatment sum of squares*, denoted ssT. Using the alternate expressions for $ssE_0$ and ssE, we have:

$$ssT = ssE_0\text{-}ssE = \sum_{i=1}^{v}\sum_{t=1}^{r_i} y_{it}^2 - n(\bar{y}_{..})^2 - \left(\sum_{i=1}^{v}\sum_{t=1}^{r_i} y_{it}^2 - \sum_{i=1}^{v} r_i(\bar{y}_{i\bullet})^2\right)$$

$$= \sum_{i=1}^{v} r_i(\bar{y}_{i\bullet})^2 - n(\bar{y}_{..})^2$$

$$= \sum_{i=1}^{v} \frac{(y_{i\bullet})^2}{r_i} - \frac{(y_{..})^2}{n} \quad \text{(using definitions)}$$

$$= \sum_{i=1}^{v} r_i(\bar{y}_{i\bullet} - \bar{y}_{..})^2 \quad \text{(possible homework)}$$

This last expression can be considered as a "between treatments" sum of squares --- we are comparing each treatment sample mean $\bar{y}_{i\bullet}$ with the grand (overall) mean $\bar{y}_{..}$. By contrast, our denominator, ssE = $\sum_{i=1}^{v}\sum_{t=1}^{r_i}\left(y_{it}-\bar{y}_i\right)^2$ is a "within treatments" sum of squares: it compares each value with the mean for the treatment group from which the value was obtained.

Using the model assumptions, it can be proved that:

- $ssE/\sigma^2 \sim \chi^2(n - v)$
- If $H_0$ is true, $ssT/\sigma^2 \sim \chi^2(v - 1)$
- If $H_0$ is true, then ssT and ssE are independent.

Thus, <u>if</u> $H_0$ is true,

$$\frac{ssT/\sigma^2(v-1)}{ssE/\sigma^2(n-v)} \sim F_{v-1,n-v}.$$

Now $\frac{ssT/\sigma^2(v-1)}{ssE/\sigma^2(n-v)}$ simplifies to $\frac{ssT/(v-1)}{ssE/(n-v)}$, which we can calculate from our sample.

We originally wanted to test ssT/ssE, but $\frac{ssT/(v-1)}{ssE/(n-v)}$ is just a constant multiple of ssT/ssE, so is good enough for our purposes: $\frac{ssT/(v-1)}{ssE/(n-v)}$ will be unusually large exactly when ssT/ssE is unusually large. Thus, we can use an F test, with test statistic $\frac{ssT/(v-1)}{ssE/(n-v)}$, to test our hypothesis.

*Note*: We can look at ssT/(v-1) and ssE/(n-v) as we did in the equal-variance, two-sample t-test: ssE/(n-v) is a pooled estimate of the common variance $\sigma^2$, and if $H_0$ is true, then ssT/(v - 1) can be regarded as an estimate of $\sigma^2$.

*Notation*: ssT/(v-1) is called msT (*mean square for treatment* or *treatment mean square* and ssE/(n-v) is called msE (*mean square for error* or *error mean square*). So the test statistic is F = msT/msE.