INFERENCE FOR ONE-WAY ANOVA

To test equality of means for different treatments/levels, we can use the null hypothesis

\[ H_0: \mu_1 = \mu_2 = \ldots = \mu_v \]

Rephrase:

1. In terms of effects: _________________________________

2. In terms of differences of effects: _________________________________

3. In terms of contrasts \( \tau_i - \overline{\tau} \), where \( \overline{\tau} = \frac{1}{v} \sum_{i=1}^{v} \tau_i \): ________________________________

The *treatment degrees of freedom* is the minimum number of equations needed to state the null hypothesis, in other words ________________.

Alternate hypothesis: \( H_a: \) ______________________________

Idea of the test: Compare ssE under the *full* model (with all parameters) with the error sum of squares ssE\(_0\) under the *reduced* model -- i.e., the one assuming \( H_0 \) is true.

To calculate ssE\(_0\): If \( H_0 \) is true, let \( \tau \) be the common value of the \( \tau_i \)'s. Then the reduced model is

- \( Y_{it} = \mu + \tau + \varepsilon_{it}^0 \)
- \( \varepsilon_{it}^0 \sim N(0, \sigma^2) \)
- the \( \varepsilon_{it}^0 \)'s are independent,

where \( \varepsilon_{it}^0 \) denotes the it\(^{th}\) error in the reduced model.

To find ssE\(_0\), we use least squares to minimize \( g(m) = \sum_{i=1}^{v} \sum_{t=1}^{r_i} (y_{it} - m)^2 \):

\[ g'(m) = \sum_{i=1}^{v} \sum_{t=1}^{r_i} 2(-1)(y_{it} - m) = 0, \]

which yields estimate \( \overline{y}.. \) for \( \mu + \tau \) -- that is, the least squares estimate of \( \mu + \tau \) is \( (\mu + \tau)^{\hat{}} = \overline{y}.. \). (By abuse of notation, we call this \( \hat{\mu} + \hat{\tau} \)). So

\[ \text{ssE}_0 = \sum_{i=1}^{v} \sum_{t=1}^{r_i} (y_{it} - \overline{y}..)^2, \]

which can be shown (proof might be homework) to equal \( \sum_{i=1}^{v} \sum_{t=1}^{r_i} y_{it}^2 - n(\overline{y}..)^2 \)
Note that \(ssE\) and \(ssE_0\) can be considered as minimizing the same expression, but over different sets: \(ssE\) minimizes \(\sum_{i=1}^{v} \sum_{t=1}^{r} (y_{it} - m - t_i)^2\) over the set of all \(v + 1\)-tuples \((m, t_1, t_2, \ldots, t_v)\), whereas \(ssE_0\) can be considered as minimizing the same expression over the subset where all \(t_i\)'s are zero. Thus \(ssE_0\) must be at least as large as \(ssE\): \(ssE_0 \geq ssE\).

However, if \(H_0\) is true, then \(ssE\) and \(ssE_0\) should be about the same. This suggests the idea of using the ratio \((ssE_0 - ssE)/ssE\) as a test statistic for the null hypothesis: If \(H_0\) is true, this ratio should be small; so an unusually large ratio would be reason to reject the null hypothesis.

The difference \(ssE_0 - ssE\) is called the *sum of squares for treatment*, or *treatment sum of squares*, denoted \(ssT\). Using the alternate expressions for \(ssE_0\) and \(ssE\), we have:

\[
ssT = ssE_0 - ssE = \sum_{i=1}^{v} \sum_{t=1}^{r} y_{it}^2 - n(\bar{y}_{..})^2 - \left( \sum_{i=1}^{v} \sum_{t=1}^{r} y_{it}^2 - \sum_{i=1}^{v} r_i (\bar{y}_{i.})^2 \right)
\]

\[
= \sum_{i=1}^{v} r_i (\bar{y}_{i.})^2 - n(\bar{y}_{..})^2
\]

\[
= \sum_{i=1}^{v} \frac{(y_{i.})^2}{r_i} - \frac{(y_{..})^2}{n} \quad \text{(using definitions)}
\]

\[
= \sum_{i=1}^{v} r_i (\bar{y}_{i.} - \bar{y}_{..})^2 \quad \text{(possible homework)}
\]

This last expression can be considered as a "between treatments" sum of squares --- we are comparing each treatment sample mean \(\bar{y}_{i.}\) with the grand (overall) mean \(\bar{y}_{..}\). By contrast, our denominator, \(ssE = \sum_{i=1}^{v} \sum_{t=1}^{r} (y_{it} - \bar{y}_{i.})^2\) is a "within treatments" sum of squares: it compares each value with the mean for the treatment group from which the value was obtained.

Using the model assumptions, it can be proved that:

- \(ssE/\sigma^2 \sim \chi^2(n - v)\)
- If \(H_0\) is true, \(ssT/\sigma^2 \sim \chi^2(v - 1)\)
- If \(H_0\) is true, then \(ssT\) and \(ssE\) are independent.

Thus, if \(H_0\) is true,

\[
\frac{ssT/\sigma^2(v - 1)}{ssE/\sigma^2(n - v)} \sim F_{v-1,n-v}.
\]
Now \( \frac{ssT}{ssE} \frac{\sigma^2 (v - 1)}{\sigma^2 (n - v)} \) simplifies to \( \frac{ssT}{ssE} \frac{(v - 1)}{(n - v)} \), which we can calculate from our sample.

We originally wanted to test \( ssT/ssE \), but \( \frac{ssT}{ssE} \frac{(v - 1)}{(n - v)} \) is just a constant multiple of

\( ssT/ssE \), so is good enough for our purposes: \( \frac{ssT}{ssE} \frac{(v - 1)}{(n - v)} \) will be unusually large exactly when \( ssT/ssE \) is unusually large. Thus, we can use an F test, with test statistic

\( \frac{ssT}{ssE} \frac{(v - 1)}{(n - v)} \), to test our hypothesis.

**Note:** We can look at \( ssT/(v-1) \) and \( ssE/(n-v) \) as we did in the equal-variance, two-sample t-test: \( ssE/(n-v) \) is a pooled estimate of the common variance \( \sigma^2 \), and if \( H_0 \) is true, then \( ssT/(v - 1) \) can be regarded as an estimate of \( \sigma^2 \).

**Notation:** \( ssT/(v-1) \) is called msT (mean square for treatment or treatment mean square and \( ssE/(n-v) \) is called msE (mean square for error or error mean square). So the test statistic is \( F = msT/msE \).