INFERENCE FOR SIMPLE OLS

Model Assumptions ("The" Simple Linear Regression Model Version IV):
(We consider \( x_1, \ldots, x_n \) as fixed.)

1. \( E(Y|x) = \eta_0 + \eta_1 x \) \hspace{1cm} (linear mean function)
2. \( \text{Var}(Y|x) = \sigma^2 \) (Equivalently, \( \text{Var}(e|x) = \sigma^2 \)) \hspace{1cm} (constant variance)
3. \( y_1, \ldots, y_n \) are independent observations. \hspace{1cm} (independence)
4. (NEW) \( Y|x \) is normal for each \( x \) \hspace{1cm} (normality)

(1) + (2) + (4) can be summarized as:

\[
Y|x \sim N(\eta_0 + \eta_1 x, \sigma^2)
\]

Recall: \( e|x = Y|x - E(Y|x) \)

So:

\[
e|x \sim N(0, \sigma^2)
\]

i.e., all errors have the same distribution -- so we just say \( e \) instead of \( e|x \).

Since \( \hat{\eta}_0 \) and \( \hat{\eta}_1 \) are linear combinations of the \( Y|x \)'s, (3) + (4) imply that \( \hat{\eta}_0 \) and \( \hat{\eta}_1 \) (that is, their sampling distributions) are normally distributed. Recalling that

\[
E(\hat{\eta}_1) = \eta_1, \hspace{1cm} \text{Var}(\hat{\eta}_1) = \frac{\sigma^2}{SXX} \hspace{1cm} E(\hat{\eta}_0) = \eta_0 \hspace{1cm} \text{Var}(\hat{\eta}_0) = \sigma^2 \left( \frac{1}{n} + \frac{\bar{x}^2}{SXX} \right)
\]

We have

\[
\hat{\eta}_1 \sim \hat{\eta}_0 \sim
\]

Look more at \( \hat{\eta}_1 \): We can standardize to get

\[
\frac{\hat{\eta}_1 - \eta_1}{\sqrt{\sigma^2/SXX}} \sim N(0,1)
\]

But we don't know \( \sigma^2 \), so need to approximate it by \( \hat{\sigma}^2 \) -- in other words approximate \( \text{Var}(\hat{\eta}_1) \) by \( \text{Var}(\hat{\eta}_1) = \text{s.e.}(\hat{\eta}_1)^2 = \frac{\hat{\sigma}^2}{SXX} \). Thus we want to use \( \frac{\hat{\eta}_1 - \eta_1}{\sqrt{\hat{\sigma}^2/SXX}} \). But we can't expect this to be normal, too. However,
\[
\frac{\hat{\eta} - \eta}{\sqrt{\hat{\sigma}^2 / SXX}} = \frac{\hat{\eta} - \eta}{\sqrt{\hat{\sigma}^2 / SXX}} \\
(*) \\
\frac{\hat{\eta} - \eta}{\sqrt{\hat{\sigma}^2 / SXX}}
\]

The numerator of the last fraction is normal (in fact, standard normal), as noted above.

Facts: (Proofs omitted)

a. \((n-2) \frac{\hat{\sigma}^2}{\sigma^2}\) has a \(\chi^2\) distribution with \(n-2\) degrees of freedom

Notation: \((n-2) \frac{\hat{\sigma}^2}{\sigma^2} \sim \chi^2(n-2)\)

b. \((n-2) \frac{\hat{\sigma}^2}{\sigma^2}\) is independent of \(\hat{\eta} - \eta\) (hence independent of the numerator in \((*)\))

Comments on distributions:

1. A \(\chi^2(k)\) distribution is defined as the distribution of a random variable which is a sum of squares of \(k\) independent standard normal random variables.

[Comment: Recall that \(\hat{\sigma}^2 = \frac{1}{n-2} \text{RSS}\), so \((n-2) \frac{\hat{\sigma}^2}{\sigma^2} = \frac{\text{RSS}}{\sigma^2} = \sum \left(\frac{\hat{e}_i}{\sigma}\right)^2\) is a sum of \(n\) squares; the fact quoted above says that it can also be expressed as a sum of \(n-2\) squares of independent standard normal random variables.]

2. A \(t\)-distribution with \(k\) degrees of freedom is defined as the distribution of a random variable of the form \(\frac{Z}{\sqrt{U/k}}\) where

- \(Z \sim N(0,1)\)
- \(U \sim \chi^2(k)\)
- \(Z\) and \(U\) are independent.

In the fraction \((*)\) above, take

\[
U = (n-2) \frac{\hat{\sigma}^2}{\sigma^2} \sim \chi^2(n-2) \\
Z = \frac{\hat{\eta} - \eta}{\sqrt{\hat{\sigma}^2 / SXX}} \sim N(0,1)
\]
Thus: \[
\frac{\hat{\eta} - \eta}{\sqrt{\frac{s^2}{SXX}}} \sim t(n-2),
\]
so we can do inference on $\eta_1$, using $t = \frac{\hat{\eta} - \eta}{\sqrt{\frac{s^2}{SXX}}}$ as our test statistic.

**Inference on $\eta_0$**

With the same assumptions, it can be shown in an analogous manner (details omitted) that
\[
\frac{\hat{\eta}_0 - \eta_0}{s.e.(\hat{\eta}_0)} \sim t(n-2),
\]
so we can use this statistic to do inference on $\eta_0$. 