CONDITIONAL MEANS AND VARIANCES, CONTINUED:
CONDITIONAL VARIANCES

Marginal Variance: The definition of the (population) (marginal) variance of a random variable $Y$ is

$$\text{Var}(Y) = E((Y - E(Y))^2)$$

What does this say in words (and pictures)?

There is another formula for $\text{Var}(Y)$ that is sometimes useful in computing variances or proving things about them. It can be obtained by multiplying out the squared expression in the definition:

$$\text{Var}(Y) = E((Y - E(Y))^2) = E(Y^2 - 2YE(Y) + [E(Y)]^2)$$

(Fill in details, and say the final result in words!)

Conditional Variance: Similarly, if we are considering a conditional distribution $Y|X$, we define the conditional variance

$$\text{Var}(Y|X) = \text{Variance of the conditional distribution } Y|X$$

$$= E((Y - E(Y|X))^2 | X)$$

(Note that both expected values here are conditional expected values.)

What does this say in words (and pictures)?

Exercise: Derive another formula for the conditional variance, analogous to the second formula for the marginal variance. (And say it in words!)

Conditional Variance as a Random Variable: As with $E(Y|X)$, we can consider $\text{Var}(Y|X)$ as a random variable. For example, if $Y = \text{height}$ and $X = \text{sex}$ for persons in a certain population, then $\text{Var}(\text{height} | \text{sex})$ is the variable which assigns to each person in the population the variance of height for that person's sex.

Expected Value of the Conditional Variance: Since $\text{Var}(Y|X)$ is a random variable, we can talk about its expected value. Using the formula $\text{Var}(Y|X) = E(Y^2|X) - [E(Y|X)]^2$, we have

$$E(\text{Var}(Y|X)) = E(E(Y^2|X)) - E([E(Y|X)]^2)$$
We have already seen that the expected value of the conditional expectation of a random variable is the expected value of the original random variable, so applying this to $Y^2$ gives

\[ (*) \quad E(\text{Var}(Y|X)) = E(Y^2) - E([E(Y|X)]^2) \]

**Variance of the Conditional Expected Value:** For what comes next, we will need to consider the variance of the conditional expected value. Using the second formula for variance, we have

\[ \text{Var}(E(Y|X)) = E([E(Y|X)]^2) - [E(E(Y|X))]^2 \]

Since $E(E(Y|X)) = E(Y)$, this gives

\[ (**) \quad \text{Var}(E(Y|X)) = E([E(Y|X)]^2) - [E(Y)]^2. \]

**Putting It Together:**

Note that (*) and (**) both contain the term $E([E(Y|X)]^2)$, but with opposite signs. So adding them gives:

\[ E(\text{Var}(Y|X)) + \text{Var}(E(Y|X)) = E(Y^2) - [E(Y)]^2, \]

which is just $\text{Var}(Y)$. In other words,

\[ (***) \quad \text{Var}(Y) = E(\text{Var}(Y|X)) + \text{Var}(E(Y|X)). \]

In words: The marginal variance is the sum of the expected value of the conditional variance and the variance of the conditional means.

**Consequences:**

I) This says that two things contribute to the marginal (overall) variance: the expected value of the conditional variance, and the variance of the conditional means. (See Exercise) Moreover, $\text{Var}(Y) = E(\text{Var}(Y|X))$ if and only if $\text{Var}(E(Y|X)) = 0$. What would this say about $E(Y|X)$?

II) Since variances are always non-negative, (***) implies

\[ \text{Var}(Y) \geq E(\text{Var}(Y|X)). \]

III) Since $\text{Var}(Y|X) \geq 0$, $E(\text{Var}(Y|X))$ must also be $\geq 0$. (Why?). Thus (***) implies

\[ \text{Var}(Y) \geq \text{Var}(E(Y|X)). \]
Moreover, $\text{Var}(Y) = \text{Var}(E(Y|X))$ if and only if $E(\text{Var}(Y|X)) = 0$. What would this imply about $\text{Var}(Y|X)$ and about the relationship between $Y$ and $X$?

IV) Another perspective on (***), (cf. Textbook, pp. 36 - 37):

Note that:

i) $E(\text{Var}(Y|X))$ is a weighted average of $\text{Var}(Y|X)$

ii) $\text{Var}(E(Y|X)) = E(\{E(Y|X) - E(E(Y|X))\}^2)$
    
    $= E(\{E(Y|X) - (E(Y))^2\}$,

    which is a weighted average of $[E(Y|X) - (E(Y))^2$.

Thus, (***), says that $\text{Var}(Y)$ is a weighted mean of $\text{Var}(Y|X)$ plus a weighted mean of $[E(Y|X) - (E(Y))^2$ (and is a weighted mean of $\text{Var}(Y|X)$ if and only if all conditional expected values $E(Y|X)$ are equal to the marginal expected value $E(Y)$.)
EXERCISE: What contributes most to \( \text{Var}(Y) \): \( \text{Var}(E(Y | X)) \) or \( E(\text{Var}(Y | X)) \)?

A.

B.

C.