ASYMPTOTIC FLOCKING DYNAMICS OF CUCKER-SMALE PARTICLES IMMERSED IN COMPRESSIBLE FLUIDS

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Abstract. We propose a coupled system for the interaction between Cucker-Smale flocking particles and viscous compressible fluids, and present a global existence theory and time-asymptotic behavior for the proposed model in the spatial periodic domain $\mathbb{T}^3$. Our model consists of the kinetic Cucker-Smale model for flocking particles and the isentropic compressible Navier-Stokes equations for fluids, and these two models are coupled through a drag force, which is responsible for the asymptotic alignment between particles and fluid. For the asymptotic flocking behavior, we explicitly construct a Lyapunov functional measuring the deviation from the asymptotic flocking states. For a large viscosity and small initial data, we show that the velocities of Cucker-Smale particles and fluids are asymptotically aligned to the common velocity.

1. Introduction. The purpose of this paper is to present a global existence theory for a coupled Cucker-Smale-Navier-Stokes system describing the interaction of Cucker-Smale flocking particles and the isentropic compressible Navier-Stokes equations. The interaction problem of particles and fluid is a long-standing topic in mathematical fluid mechanics, and attracts considerable attention due to the engineering applications. When the flocking particles are surrounded by the viscous gas, the force per unit mass exerted on a flocking particle by the surrounding...
gas will come from several effects, e.g., skin friction, separation drag, gravity and body forces, rotation of the particle with respect to the gas, pressure gradient in the gas, etc (see [38]). In this paper, we are interested in the dynamic behavior of an ensemble of Cucker-Smale flocking particles interacting with the gas through the drag force. The use of drag force in the particle-fluid interaction problem can be found in previous literature [1, 2, 3, 6, 32]. Let \( f = f(x, \xi, t) \) be the one-particle distribution function of the Cucker-Smale (in short, C-S) flocking particles at the phase-space position \( (x, \xi) \in \mathbb{T}^3 \times \mathbb{R}^3 \) and \( \rho = \rho(x, t) \), \( u = u(x, t) \) be the local mass density and bulk velocity of the isentropic compressible fluid, respectively. Then, our proposed Cucker-Smale-Navier-Stokes system reads as

\[
\begin{align*}
\partial_t f + \xi \cdot \nabla_x f + \nabla_x \cdot (F_a(f) + F_d) &= 0, \quad (x, \xi) \in \mathbb{T}^3 \times \mathbb{R}^3, \quad t > 0, \\
\partial_t \rho + \text{div}_x (\rho u) &= 0, \quad p = \kappa \rho^\gamma, \quad \gamma > 1, \\
\partial_t (\rho u) + \text{div}_x (\rho u \otimes u) + \nabla_x p - \mu \Delta_x u - (\mu + \lambda) \nabla_x \text{div} u &= -K_c \int_{\mathbb{R}^3} (u(x, t) - \xi) f d\xi, \\
\end{align*}
\]

subject to initial data:

\[
(f(x, \xi, 0), \rho(x, 0), u(x, 0)) = (f_0(x, \xi), \rho_0(x), u_0(x)), \quad (x, \xi) \in \mathbb{T}^3 \times \mathbb{R}^3.
\]  

Here \( \gamma > 1 \) is an adiabatic constant, viscosity coefficients \( \mu \) and \( \lambda \) satisfy the physical requirements \( \mu > 0 \) and \( 2\mu + 3\lambda \geq 0 \), and \( F_a \) and \( F_d \) represent the alignment (flocking) force and the drag force per unit mass, respectively:

\[
\begin{align*}
F_a(f)(x, \xi, t) &= -K_f \int_{\mathbb{T}^3 \times \mathbb{R}^3} \psi(x, y)(\xi - \xi_*) f(y, \xi_*, t) d\xi_* dy, \\
F_d(x, \xi, t) &= K_c (u(x, t) - \xi).
\end{align*}
\]

The kernel function \( \psi : \mathbb{T}^3 \times \mathbb{T}^3 \to \mathbb{R}_+ \) is an \( \mathcal{C}^1 \)-function that satisfies the conditions of symmetry, and boundedness:

\[
\psi(x, y) = \psi(y, x), \quad \psi_m \leq \psi(x, y) \leq \psi_M.
\]

Here \( e_i \) is the canonical unit vector, and \( \psi_m \) and \( \psi_M \) are positive constants. Throughout the paper, without loss of generality, we assume

\[
K_f = 1, \quad K_c = 1, \quad \kappa = 1,
\]

so that the pressure law becomes \( p(\rho) = \rho^\gamma \). It should be noted that the first equation (1) is the kinetic C-S model [12] derived from the C-S particle model in [20, 21], and the second system (1) represents the isentropic compressible Navier-Stokes equations with the drag force. In this direction, the first approach for the coupling of particles and fluids was proposed by Hamdache [22]. He studied the global existence of the unsteady Stokes system coupled with the Vlasov equation in a bounded domain with the reflection boundary conditions. For this coupled Vlasov-Stokes system, he constructed a weak solution and showed the large-time behavior. Later, Hamdache’s work was extended to the Vlasov-Navier-Stokes system by Boudin et al. [6], where the global existence of weak solutions was studied. In [8], Chae et al. studied the global existence of weak and classical solutions for the Vlasov-Fokker-Planck equation coupled with the incompressible Navier-Stokes equations. In [32], Mellet and Vasseur proved the existence of global weak solutions to the coupled system of Vlasov-Fokker-Planck and compressible Navier-Stokes equations in a bounded domain with Dirichlet or reflection boundary conditions. We also refer the reader to other related articles [18, 19]. For the case when the fluid
is inviscid, Baranger and Desvillettes established the local existence of solutions to the compressible Vlasov-Euler equations \[3\]. Recently, a new coupled model for the interactions between flocking particles and fluids was proposed by Bae \[1\]. In \[1, 2\], the authors considered the incompressible Navier-Stokes equations for the fluid part, and they obtained the global existence of weak and strong solutions. They also provided the emergence of alignment (flocking) time-asymptotically for any smooth solutions, when the viscosity is sufficiently large. For more details on the coupled C-S flocking particles and incompressible viscous fluid models, we refer the reader to \[1\] and references therein. In this paper, we continue the study begun in \[1\] on isentropic compressible viscous fluid.

The main results of this paper are two-fold. First, we present a unique global solvability of strong solutions to the coupled system (1) - (2) for small initial data. We use Schauder’s fixed-point arguments to obtain the solvability of strong solutions. Second, we present an asymptotic flocking estimate by introducing a Lyapunov functional that measures the asymptotic alignment of particle velocity and fluid velocity; the fluid density also converges to the constant average density exponentially fast.

The rest of this paper is organized as follows. In Section 2, we briefly discuss our model (1) and present the main results for global solvability and asymptotic behavior. In Section 3, we present a global existence of strong solutions to the system (1). In Section 4, we study the asymptotic flocking estimate using the Lyapunov functional approach that measures the differences of velocities between particles and fluid. Finally, Section 5 is devoted to a summary of the main results. In Appendix A, we present a rather tedious lower bound estimate of interaction production rate.

**Notations.** Throughout this paper, we set \( T \in (0, \infty] \), let \( C \) be a generic constant, which may differ from line to line, and does not depend on \( T \). For a given \( t \in [0, T) \), we set

\[
P(t) := \{\xi \in \mathbb{R}^3 : \exists (x, \xi) \in T^3 \times \mathbb{R}^3 \text{ such that } f(x, \xi, t) \neq 0\},
\]

\[
\eta(t) := \max\{|\xi| : \xi \in P(t)\},
\]

\[
M_p(f(t)) := \int_{T^3 \times \mathbb{R}^3} |\xi|^p f d\xi dx, \quad p \in \mathbb{Z}_+ \cup \{0\},
\]

For the notational simplicity, we set several norms and drop \( x \)-dependence of differential operators \( \nabla_x, \Delta_x \):

\[
\nabla := \nabla_x, \quad \Delta := \Delta_x, \quad \|v\|_{L^r(0,T,W^{k,p})} := \|v\|_{L^r(0,T;W^{k,p}(T^3))},
\]

\[
\|v(t)\|_{W^{k,p}} := \|v(t)\|_{W^{k,p}(U)}, \quad U \text{ replace either } T^3 \text{ or } T^3 \times \mathbb{R}^3,
\]

\[
\|v\|_{W^{k,\infty}} := \|v\|_{W^{k,\infty}(V)}, \quad V \text{ replace either } T^3 \times [0, T] \text{ or } T^3 \times \mathbb{R}^3 \times [0, T],
\]

2. Description of model and main results. In this section, we present the main results and discuss the C-S flocking particles and isentropic compressible Navier-Stokes equations.

2.1. Description of the model. We briefly review the relevant existence theory and decay estimates of the kinetic C-S flocking model and isentropic compressible Navier-Stokes equations. We first discuss the kinetic C-S model. In \[20, 21\], Ha and his collaborators derived a dissipative kinetic Vlasov type model and rigorously established the mean-field limit from the original Cucker-Smale model.
\[
\frac{\partial f}{\partial t} + \nabla f \cdot \nabla \xi + \nabla \cdot [F_a(f)f] = 0, \quad (x,\xi,t) \in T^d \times \mathbb{R}^d \times \mathbb{R}_+,
\]

where

\[
F_a(f)(x,\xi,t) := K_f \int_{\mathbb{R}^d} \psi_{cs}(|y-x|)(\xi_* - \xi) f\xi_*dy,
\]

and \(K_f > 0\) is a positive flocking coupling strength and \(\psi_{cs}(s)\) is given by

\[
\psi_{cs}(s) = \frac{1}{(1 + s^2)^\beta}, \quad \beta \geq 0.
\]

Ha and Tadmor obtained the global existence of smooth solutions to the system (4) with smooth compactly supported initial data without any assumptions about the size of the data, and they also showed the decay of fluctuations, \(E_f(t)\), around the mean bulk velocity \(\xi_c\):

\[
E_f(t) \leq C E_f(0) \times \begin{cases} 
  e^{-ct^{1-4\beta}} & 0 \leq \beta < \frac{1}{4}, \\
  (1 + t)^{-c'} & \beta = \frac{1}{4},
\end{cases}
\]

where

\[
E_f(t) := \int_{\mathbb{R}^d} |\xi - \xi_c|^2 f\xi dx, \quad \xi_c(t) := \frac{\int_{\mathbb{R}^d} \xi f\xi dx}{\int_{\mathbb{R}^d} f\xi dx}
\]

and \(C, c\) and \(c'\) are positive constants. Later, the estimate (5) was extended to the consistent estimate, as in the particle C-S model proposed by Carrillo et al. in [7].

On the other hand, for a fluid model, we consider the isentropic compressible Navier-Stokes equations:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \text{div}(\rho u) &= 0, \\
\frac{\partial (\rho u)}{\partial t} + \text{div}(\rho u \otimes u) + \nabla p &= \mu \Delta u + (\mu + \lambda) \nabla \text{div} u, \\
p(\rho) &= \kappa \rho^\gamma \quad (\kappa > 0, \gamma > 1).
\end{align*}
\]

The global well-posedness of weak solutions to (6) on the whole domain and bounded domains has been studied in [30, 31, 36, 37]. In [29], Lions proved the general result of weak solutions to the multidimensional isentropic compressible Navier-Stokes equations with large initial data. Later, Lions’ theory was developed further by several researchers, e.g., [14, 16]. In [10, 11], Choe et al. established the local existence of strong solutions. When initial density \(\rho_0\) is bounded away from zero or some positive fixed density, we refer to [13, 23, 24, 30, 31] for the local and global existence of classical solutions to the isentropic compressible Navier-Stokes equations (6). In [9], Cho also obtained the global existence of smooth radially symmetric solutions on annular or exterior domains.

For the large time behavior of solutions to the system (6), Matsumura and Nishida [30, 31] provided the \(H^s\) global existence and time-decay estimates of strong solutions in the whole space \(\mathbb{R}^3\), and later the optimal \(L^p\) \((p \geq 2)\) decay rate was obtained by Ponce in [33]. For the multi-dimensional half space or exterior domain, the large time behavior of solutions was also studied in [25, 26, 27, 28]. From the previous results, the optimal \(L^2\) time-decay estimates in three-dimension are given as

\[
\|(\rho - \bar{\rho}, u)(t)\|_{L^2(\mathbb{R}^3)} \leq C(1 + t)^{-\frac{3}{4}},
\]

with \((\bar{\rho}, 0)\) the constant state. Recently, under the assumption that \(\rho\) has an upper bound, Fang et al. [15] showed that the weak solutions decay exponentially to the equilibrium state in \(L^2\) norm. In addition, studies have been conducted on
the large time-decay estimates for the solutions to (6) when there are exterior or external potential forces [34, 35].

So far, we briefly described the kinetic C-S model and compressible Navier-Stokes equations separately. We now consider a situation where an ensemble of self-propelled particles (e.g., microscopic unmanned aerial vehicles) moves in ambient compressible viscous fluids. Then, the physical situation that we have in mind is to make particles and fluid align together. In this case, such modeling is reduced to the coupling of two systems for particles and fluids so that the resulting combined system exhibit a velocity flocking. Of course, in real situations such as the flocking of birds floating on the air, the flocking velocity of birds may not be the same as the bulk velocity of air, unless birds are simply drifted by the air. Unfortunately, our system (1) does not describe this situation where other mechanisms mentioned in the first paragraph of Introduction need to be added, so this is beyond the scope of our present paper. For the flocking mechanism between particle system and fluids, we employ the drag force $u(x, t) - \xi$ in (3) as in previous literature [1, 2, 3, 6, 18, 19, 38] to propose a coupled flocking particle-fluid system:

$$\partial_t f + \xi \cdot \nabla_x f + \nabla \cdot (F_a(f) f + F_d f) = 0, \quad (x, \xi) \in \mathbb{T}^3 \times \mathbb{R}^3, \quad t > 0,$$

$$\partial_t \rho + \text{div}_x (\rho u) = 0, \quad p = \kappa \rho^\gamma, \quad \gamma > 1,$$

$$\partial_t (\rho u) + \text{div}_x (\rho u \otimes u) + \nabla_x p - \mu \Delta_x u - (\mu + \lambda) \nabla_x \text{div} u = - \int_{\mathbb{R}^3} (u(x, t) - \xi) f d\xi,$$

(7)

We next provide a priori energy estimates for smooth solutions to the coupled kinetic-fluid system (7).

**Lemma 2.1.** The following relation holds.

$$\int_{\mathbb{T}^3} u \cdot \nabla p dx = \frac{1}{\gamma - 1} \frac{d}{dt} \int_{\mathbb{T}^3} p dx.$$

**Proof.** We use the constitutive relation $p = \rho^\gamma$ to obtain

$$\int_{\mathbb{T}^3} u \cdot \nabla p dx = \int_{\mathbb{T}^3} u \cdot \nabla (\rho^\gamma) dx = \gamma \int_{\mathbb{T}^3} u \cdot \rho^{\gamma - 1} \nabla \rho dx$$

$$= -\gamma \int_{\mathbb{T}^3} \rho^{\gamma - 1} (\rho_t + \rho \nabla \cdot u) dx$$

$$= -\frac{d}{dt} \int_{\mathbb{T}^3} \rho^\gamma dx - \gamma \int_{\mathbb{T}^3} \rho^\gamma \nabla \cdot u dx = -\frac{d}{dt} \int_{\mathbb{T}^3} \rho^\gamma dx + \gamma \int_{\mathbb{T}^3} \nabla p \cdot u dx.$$

Thus, we have

$$\int_{\mathbb{T}^3} u \cdot \nabla p dx = \frac{1}{\gamma - 1} \frac{d}{dt} \int_{\mathbb{T}^3} \rho^\gamma dx.$$

Since $p = \rho^\gamma$, we have the desired result. \qed

**Proposition 1.** Let $[f, \rho, u]$ be a smooth solution to (1) - (2). Then, we have the following balanced laws:

(i) $\frac{d}{dt} \int_{\mathbb{T}^3 \times \mathbb{R}^3} f d\xi dx = 0$ and $\frac{d}{dt} \int_{\mathbb{T}^3} \rho dx = 0$.

(ii) $\frac{d}{dt} \left( \int_{\mathbb{T}^3 \times \mathbb{R}^3} \xi f d\xi dx + \int_{\mathbb{T}^3} \rho u dx \right) = 0$, 
\((iii) \quad \frac{d}{dt} \left( \int_{\mathbb{T}^3 \times \mathbb{R}^3} \frac{|\xi|^2}{2} f d\xi dx + \int_{\mathbb{T}^3} \frac{1}{2} \rho |u|^2 + \frac{\rho p}{\gamma - 1} dx \right)
\quad \left(\right. 
\quad + \int_{\mathbb{T}^3} \mu |\nabla u|^2 + (\lambda + \mu) (\nabla \cdot u)^2 dx 
\quad = - \frac{1}{2} \int_{\mathbb{T}^6 \times \mathbb{R}^6} \psi(x, y)|\xi - \xi_s|^2 f(y, \xi_s) f(x, \xi) d\xi_s dy d\xi dx 
\quad \left. - \int_{\mathbb{T}^3 \times \mathbb{R}^3} |u - \xi|^2 f d\xi dx. \right)\)

Proof. (i) Conservation of mass follows from \((1)_1\) and \((1)_2\).

(ii) We multiply the Vlasov-type flocking equation \((1)_1\) by \(\xi\), and integrate it over \(\Omega\) to obtain
\[
\frac{d}{dt} \int_{\mathbb{T}^3 \times \mathbb{R}^3} \xi f d\xi dx = \int_{\mathbb{T}^3 \times \mathbb{R}^3} (F_a(f) + F_d) f d\xi dx 
\quad = \int_{\mathbb{T}^3 \times \mathbb{R}^3} F_d f d\xi dx 
\quad = \int_{\mathbb{T}^3 \times \mathbb{R}^3} (u - \xi) f d\xi dx,
\]
where we used the symmetry of \(\psi\) to estimate \(\int_{\mathbb{T}^3 \times \mathbb{R}^3} F_a(f) f d\xi dx = 0\).

On the other hand, we also integrate \((1)_3\) to find
\[
\frac{d}{dt} \int_{\mathbb{T}^3} \rho u dx = - \int_{\mathbb{T}^3 \times \mathbb{R}^3} (u - \xi) f d\xi dx.
\]

Then, the desired conservation of momentum follows from \((8)\) - \((9)\).

(iii) For the time-variation of particle energy, we have
\[
\frac{d}{dt} \int_{\mathbb{T}^3 \times \mathbb{R}^3} \frac{|\xi|^2}{2} f d\xi dx = \int_{\mathbb{T}^3 \times \mathbb{R}^3} \xi \cdot (F_a(f) + F_d) f d\xi dx 
\quad = - \frac{1}{2} \int_{\mathbb{T}^6 \times \mathbb{R}^6} \psi(x, y)|\xi - \xi_s|^2 f(y, \xi_s) f(x, \xi) d\xi_s dy d\xi dx 
\quad \left. + \int_{\mathbb{T}^3 \times \mathbb{R}^3} \xi \cdot (u - \xi) f d\xi dx. \right)
\]

For the fluid part, it follows from \((1)\) that
\[
\frac{d}{dt} \int_{\mathbb{T}^3} \frac{\rho}{2} |u|^2 dx 
\quad = \frac{1}{2} \int_{\mathbb{T}^3} \rho_t |u|^2 dx + \int_{\mathbb{T}^3} \rho u \cdot u dx 
\quad = - \frac{1}{2} \int_{\mathbb{T}^3} \nabla \cdot (\rho u) |u|^2 dx + \int_{\mathbb{T}^3} \rho u_t \cdot u dx 
\quad = \frac{1}{2} \int_{\mathbb{T}^3} \rho u \cdot \nabla (|u|^2) dx + \int_{\mathbb{T}^3} \rho u_t \cdot u dx 
\quad = \int_{\mathbb{T}^3} u \cdot (\rho u \cdot \nabla u + \rho u_t) dx 
\quad = - \int_{\mathbb{T}^3} u \cdot \left[ \nabla p - \mu \Delta u - (\mu + \lambda) \nabla (\nabla \cdot u) + \int_{\mathbb{R}^3} (u - \xi) f d\xi \right] dx 
\quad = - \int_{\mathbb{T}^3 \times \mathbb{R}^3} u \cdot (u - \xi) f d\xi dx.
\]
Then, we use Lemma 2.1 to obtain
\[
\frac{d}{dt} \int_{T^3} \left[ \frac{\rho}{2} |u|^2 + \frac{1}{\gamma - 1} \rho^\gamma \right] dx + \int_{T^3} \left[ \mu |\nabla u|^2 + (\lambda + \mu) (\nabla \cdot u)^2 \right] dx = - \int_{T^3 \times \mathbb{R}^3} u \cdot (u - \xi) f d\xi dx.
\]
Hence, we have the desired result by combining (10) and (11). \ \Box

\textbf{Remark 1.} From Lemma 1 (i), we obtain
\[
\int_{T^3 \times \mathbb{R}^3} f d\xi dx = \int_{T^3 \times \mathbb{R}^3} f_0 d\xi dx \quad \text{and} \quad \int_{T^3} \rho dx = \int_{T^3} \rho_0 dx, \quad t \geq 0.
\]

2.2. Main results. In this part, we briefly discuss two distinct frameworks for the global existence of strong solutions and time-asymptotic flocking to the system (1) - (2), respectively. We next describe two main results concerning the global existence of strong solutions and time-asymptotic flocking estimates to the system (1) - (2).

2.2.1. A global existence of strong solutions. We recall the definition of a strong solution to (1) - (2) as follows.

\textbf{Definition 2.2.} For a given \( T \in (0, \infty) \), let \([f, \rho, u]\) be a strong solution of (1) - (2) on the time-interval \([0, T]\) if and only if the following conditions are satisfied:

(i) \( f \in W^{1, \infty}(T^3 \times \mathbb{R}^3 \times [0, T]) \),
(ii) \( \rho \in L^\infty(0, T; H^2(T^3)) \), \( \rho_t \in L^\infty(0, T; H^1(T^3)) \),
(iii) \( u \in L^\infty(0, T; H^2(T^3)) \cap L^2(0, T; H^3(T^3)) \cap H^1(0, T; H^1(T^3)) \),
(iv) \( (f, \rho, u) \) satisfies the coupled system (1) - (3) in the distributional sense.

Our first framework \((\mathcal{G})\) concerns the global existence of strong solutions.

- \((\mathcal{G})\): Initial data \([f_0, \rho_0, u_0]\) satisfy the normalization, compact velocity support and smallness, non-vacuum conditions
  
  (i) \( \text{supp}_x(f_0(x, \cdot)) \) is bounded, \( \forall \, x \in T^3 \),
  (ii) \( \|f_0\|_{W^{1, \infty}} + \|u_0\|_{H^2} < \varepsilon_1 \), \( \|\rho_0\|_{H^2} < \varepsilon_1^{7/16} \), \( \inf_{x \in T^3} \rho_0(x) \geq \varepsilon_1^{1/2} \),

where the small positive constant \( \varepsilon_1 \) satisfies \( \varepsilon_1 e^{CT} = O(1) \) and \( C \) is a sufficiently large constant which does not depend on \( T \).

Under the framework \((\mathcal{G})\), the global existence of a unique strong solution can be guaranteed as follows.

\textbf{Theorem 2.3.} (Unique global solvability) For a given \( T \in (0, \infty) \), suppose that the assumptions in \((\mathcal{G})\) hold. Then, there exists a unique strong solution \([f, \rho, u]\) to the system (1) - (2) in the time-interval \([0, T]\).

2.2.2. Emergence of time-asymptotic flocking. For the flocking estimate, we introduce a Lyapunov functional \( \mathcal{L} \) that measures the local velocity fluctuations and the distance between local velocity averages:
\[
\mathcal{L}(t) := \frac{1}{2} \int_{T^3 \times \mathbb{R}^3} |\xi - \xi_c|^2 f(x, \xi) d\xi dx + \frac{1}{2} \int_{T^3} \rho |u - u_c|^2 dx + \int_{T^3} (\rho - \rho_c)^2 dx + |u_c - \xi_c|^2.
\]

Here, \( u_c \) and \( \xi_c \) are the velocity averages for fluids and particles, respectively, and \( \rho_c \) is the average of \( \rho_0 \):
\[
\xi_c(t) := \frac{\int_{T^3 \times \mathbb{R}^3} \xi f d\xi dx}{\int_{T^3 \times \mathbb{R}^3} f d\xi dx}, \quad u_c(t) := \int_{T^3} u dx, \quad \text{and} \quad \rho_c(t) := \int_{T^3} \rho_0 dx.
\]

(12)
In the sequel, without loss of generality, we assume \( \|f_0\|_{L^1} = 1 \), we thus rewrite \( \xi_c(t) = \int_{T^3 \times \mathbb{R}^3} f d\xi dx \). We set \( E(t) \) to be the total energy of the system:

\[
E(t) := \int_{T^3 \times \mathbb{R}^3} \|\xi\|^2 f d\xi dx + \int_{T^3} \left[ \frac{1}{2} \rho |u|^2 + \frac{p(\rho)}{\gamma - 1} \right] dx.
\]

Our second framework \( \mathcal{A} \) for the time-asymptotic alignment is a kind of \textit{a priori} setting. Let \( T \) be any positive number.

- \( \mathcal{A}_1 \): The viscosity \( \mu \) is sufficiently large to satisfy

\[
\mu > \frac{8}{3} \|\rho_p\|_{L^\infty} \quad \text{where} \quad \rho_p := \int_{\mathbb{R}^3} f d\xi.
\]

- \( \mathcal{A}_2 \): \([f, \rho, u]\) satisfy smallness and boundedness:

\[
\|\rho\|_{L^\infty} \ll 1, \quad \|u\|_{L^\infty} \ll 1, \quad \mathcal{L}(0) < \infty, \quad \mathcal{E}(0) < \infty.
\]

Our second result concerns the emergence of time-asymptotic alignment.

**Theorem 2.4** (Emergence of time-asymptotic alignment). Let \([f, \rho, u]\) be a global smooth solution to system \((1)-(2)\) satisfying the framework \( \mathcal{A} \). Then, we have the following exponential alignment between C-S particles and fluid:

\[
\mathcal{L}(t) \leq C_1 \mathcal{L}(0) \exp\{-C_2 t\}, \quad \text{for} \quad t \geq 0.
\]

Here, \( C_1 \) and \( C_2 \) are positive constants independent of \( t \).

**Remark 2.** The exponential alignment between C-S particles and fluid are mainly due to the drag force itself. So when the drag force \((3)\) are turned off \( K_c = 0 \), the flocking velocity of C-S particles will be different from the bulk velocity of fluid. In fact, even for the absence of the flocking force, \( K_f = 0 \), the exponential alignment between C-S particles and fluid will occur asymptotically, but in the presence of flocking force \( K_f > 0 \), the speed of alignment will be faster than zero flocking forcing case.

3. **Global existence of strong solutions.** In this section, we provide the global existence of the unique strong solution to system \((1)-(2)\) using Schauder’s fixed-point theorem.

3.1. **Estimates on a frozen linear system.** Consider the following linearized system of \((1)\) on \( u \):

\[
\begin{aligned}
\partial_t f + \xi \cdot \nabla f + \nabla \xi \cdot [F(f, \bar{u})] &= 0, \\
\partial_t \rho + \text{div}(\rho \bar{u}) &= 0, \\
\rho \partial_t u + \rho \bar{u} \cdot \nabla u + \nabla p + Lu &= -\int_{\mathbb{R}^3} (\bar{u} - \xi)f d\xi
\end{aligned}
\]

subject to initial data:

\[
(f(x, \xi, 0), \rho(x, 0), u(x, 0)) = (f_0(x, \xi), \rho_0(x), u_0(x)), \quad (x, \xi) \in T^3 \times \mathbb{R}^3.
\]

Here, \( Lu \) and \( F \) are given by

\[
Lu := -\mu \Delta u - (\mu + \lambda) \nabla \text{div} u \quad \text{and} \quad F(f, \bar{u}) = F_a(f) + \bar{u} - \xi.
\]
For a given \((x, \xi) \in \Omega\) at time \(t\), we define a forward particle trajectory \((x(t), \xi(t)) := (x(t; x, \xi, 0), \xi(t; x, \xi, 0))\) to be the solution of the following ODEs:

\[
\begin{align*}
\frac{dx(t)}{dt} &= \xi(t), \\
\frac{d\xi(t)}{dt} &= F_{\nu}(f)(x(t), \xi(t), t) + \bar{u}(x(t), t) - \xi(t),
\end{align*}
\]

subject to initial data:

\[
x(0) = x, \quad \xi(0) = \xi.
\]

As a first estimate on (13), we show that the uniform bound of fluid velocity \(\bar{u}\) implies the uniform boundedness of velocity support for kinetic density \(f\).

**Lemma 3.1.** Suppose that the main assumptions in \((G)\) hold, and let \(f\) be the solution of (13) with initial data \(f_0\) satisfying

\[
\eta_0 < \infty, \quad M_2(f_0) < \infty.
\]

If there exists a positive constant \(U^\infty > 0\) such that

\[
\|\bar{u}\|_{L^\infty(T^3 \times [0, T])} \leq U^\infty,
\]

we have

\[
\eta(t) \leq \eta^\infty, \quad 0 \leq t \leq T,
\]

where \(\eta^\infty\) is a positive constant defined as

\[
\eta^\infty := 3 \left(\eta_0 + \frac{\psi_M(\sqrt{M_2(f_0)} + U^\infty) + U^\infty}{\psi_m + 1}\right), \quad \eta_0 := \eta(0).
\]

**Proof.** First, it follows from (10) and (13) that

\[
\frac{d}{dt} M_2(f(t)) \leq 2 \int_{T^3 \times R^3} \xi \cdot (\bar{u} - \xi) f d\xi dx \leq 2 \|\bar{u}\|_{L^\infty} \sqrt{M_2(f(t))} - 2 M_2(f(t)).
\]

Then, we have

\[
M_2(f(t)) \leq (\sqrt{M_2(f_0)} + \|\bar{u}\|_{L^\infty})^2.
\]

For each \(i = 1, 2, 3\), it follows from (14) that

\[
\frac{d}{ds}|\xi_i(s)| = |(F_{\nu})_i(f)(x(s), \xi(s), s) + \bar{u}_i(x(s), s) - \xi_i(s)|\text{sgn}(\xi_i(s))
\]

\[
\leq \psi_M \sqrt{M_2(f(t))} + \|\bar{u}\|_{L^\infty} - (\psi_m + 1)|\xi_i(s)|
\]

\[
\leq \psi_M (\sqrt{M_2(f_0)} + U^\infty) + U^\infty - (\psi_m + 1)|\xi_i(s)|, \quad \xi = (\xi_1, \xi_2, \xi_3).
\]

Then, Gronwall’s lemma implies

\[
|\xi_i(t)| \leq |\xi_i(0)| e^{-(\psi_m + 1)t} + \frac{\psi_M (\sqrt{M_2(f_0)} + U^\infty) + U^\infty}{\psi_m + 1} (1 - e^{-(\psi_m + 1)t})
\]

\[
\leq |\xi_i(0)| + \frac{\psi_M (\sqrt{M_2(f_0)} + U^\infty) + U^\infty}{\psi_m + 1}.
\]

If we consider \(\xi(0) \in P(0)\), we have

\[
|\xi(t)| \leq \sum_{i=1}^3 |\xi_i(t)| \leq \eta^\infty.
\]
By applying the method of characteristics to (13) with (14), we have
\[ f(x(t), \xi(t), t) = f_0(x, \xi) \exp \left( \int_0^t \int_{\Omega} \psi(x(s), y)f(y, \xi_s, s)dyd\xi_s ds + 3t \right). \]
This mild form and the uniform boundedness of \( \xi(t) \) imply (15). \( \square \)

**Lemma 3.2.** Suppose that \( \bar{u} \) and initial datum \( f_0 \) satisfy
\[
(i) \, ||\bar{u}||_{L^2(0,T;H^3)} \leq U^\infty, \text{ for some positive constant } U^\infty.
(ii) \, ||f_0||_{W^{1,\infty}} < \varepsilon_1, \quad \varepsilon_1 e^{CT} = O(1),
\]
where \( C \) is a sufficiently large constant that does not depend on \( T \). Then, there exists a unique solution \( f \) to (13) satisfying
\[ ||f||_{W^{1,\infty}} \leq \varepsilon_1^{2/3}. \]

**Proof.** The proof is almost the same as that in [1]. We first introduce a nonlinear transport operator \( N \) on the phase-space \( T^3 \times \mathbb{R}^3 \) associated with (13):
\[ N := \partial_t + \xi \cdot \nabla + F(f, \bar{u}) \cdot \nabla \xi. \]
Then, it follows from (13), that
\[
N(f) = -(\nabla \xi \cdot F(f, \bar{u})) f \leq C f,
N(\partial_x f) = -\partial_x F(f, \bar{u}) \cdot \nabla \xi f - (\nabla \xi \cdot \partial_x F(f, \bar{u})) f - (\nabla \xi \cdot F(f, \bar{u})) \partial_x f
\leq C((1 + ||\nabla \bar{u}||_{L^\infty})||\nabla \xi f|| + ||f|| + ||\partial_x f||),
N(\partial_{\xi} f) = -\partial_{\xi} F(f, \bar{u}) \cdot \nabla \xi f - (\nabla \xi \cdot F(f, \bar{u})) \partial_{\xi} f
\leq C(||\partial_x f|| + ||\nabla \xi f||),
\]
where we used the estimates
\[
\nabla \xi \cdot F(f, \bar{u}) \leq 3\psi_M + 3, \quad \partial_{\xi} F(f, \bar{u}) \leq \psi_M + 1,
\partial_x F(f, \bar{u}) \leq C||\nabla \psi||_{L^\infty} + ||\nabla \bar{u}||_{L^\infty}, \quad \nabla \xi \cdot \partial_x F(f, \bar{u}) \leq 3||\nabla \psi||_{L^\infty}.
\]
We set \( \mathcal{F}(t) \) measuring the \( W^{1,\infty} \)-norm of \( f \):
\[ \mathcal{F}(t) := \sum_{0 \leq |\alpha| + |\beta| \leq 1} ||\nabla^\alpha \nabla^\beta_x f(t)||_{L^\infty}. \]
It follows from estimates (16) that
\[
\frac{d\mathcal{F}(t)}{dt} \leq C(1 + ||\nabla \bar{u}(t)||_{L^\infty})\mathcal{F}(t), \quad t \in (0,T).
\]
Thus, we have
\[ \mathcal{F}(t) \leq \mathcal{F}(0) \exp (CT + C||\nabla \bar{u}||_{L^1(0,T;L^\infty)}), \quad t \in (0,T). \]
On the other hand, note that
\[ \mathcal{F}(0) < \varepsilon_1, \quad ||\nabla \bar{u}||_{L^1(0,T;L^\infty)} \leq C\sqrt{T}||\bar{u}||_{L^2(0,T;H^3)} \leq C U^\infty \sqrt{T}. \]
Thus, we have
\[ \mathcal{F}(t) \leq \varepsilon_1 e^{CT} \leq \varepsilon_1^{2/3}. \]
Lemma 3.3. Suppose that the initial data $\rho_0$ and $\bar{u}$ satisfy

(i) $\|\rho_0\|_{L^2} < \varepsilon_1^{7/16}$, $\inf_{x \in \mathbb{T}^3} \rho_0(x) \geq \varepsilon_1^{1/2}$,

(ii) $\|\bar{u}\|_{L^\infty(0,T;H^2)}^2 + \|\bar{u}\|_{L^2(0,T;H^3)}^2 \leq \varepsilon_1^{1/7}$,

where $\varepsilon$ is a constant given in Lemma 3.2. Then, there exists a unique solution $\rho$ to (13)$_2$ satisfying

(i) $\|\rho\|_{L^\infty(0,T;H^2)} + \|\partial_t \rho\|_{L^\infty(0,T;H^1)} < C\varepsilon_1^{7/16}$,

(ii) $\rho(t, x) > 0$ for all $(t, x) \in (0, T) \times \mathbb{T}^3$,

(iii) $\|\rho\|_{L^\infty(0,T;H^2)} + \|\partial_t \rho\|_{L^\infty(0,T;H^1)} < C\varepsilon_1^{3/8}$.

Proof. (i) First, the existence and uniqueness of the linearized continuity equation (13)$_2$ are well-known. In order to obtain the regularity of $\rho$, we first multiply (13)$_2$ by $\rho$ and integrate over $\mathbb{T}^3$ to obtain

$$
\frac{1}{2} \frac{d}{dt} \|\rho\|_{L^2}^2 = \int_{\mathbb{T}^3} \rho \nabla \cdot \bar{u} dx = -\frac{1}{2} \int_{\mathbb{T}^3} (\nabla \cdot \bar{u}) \rho^2 dx \\
\leq \frac{1}{2} \|\nabla \cdot \bar{u}\|_{L^\infty} \|\rho\|_{L^2}^2 \leq C \|\nabla \bar{u}\|_{L^\infty} \|\rho\|_{L^2}^2.
$$

We next take the gradient of (13)$_2$, multiply by $\nabla \rho$, and integrate over $\mathbb{T}^3$ to obtain

$$
\frac{1}{2} \frac{d}{dt} \|\nabla \rho\|_{L^2}^2 = -\int_{\mathbb{T}^3} \nabla (\bar{u} \cdot \nabla \rho) \cdot \nabla \rho dx - \int_{\mathbb{T}^3} \nabla (\rho \nabla \cdot \bar{u}) \cdot \nabla \rho dx \\
:= I_{31} + I_{32}.
$$

We next estimate $I_{31}$ as

$$
I_{31} \leq \int_{\mathbb{T}^3} |\nabla \bar{u}| |\nabla \rho|^2 dx + \int_{\mathbb{T}^3} (\bar{u} \cdot \nabla) \nabla \rho \cdot \nabla \rho dx \\
= \int_{\mathbb{T}^3} |\nabla \bar{u}| |\nabla \rho|^2 dx - \frac{1}{2} \int_{\mathbb{T}^3} \nabla \cdot \bar{u} |\nabla \rho|^2 dx \\
\leq 2 \int_{\mathbb{T}^3} |\nabla \bar{u}| |\nabla \rho|^2 dx \leq C \|\nabla \bar{u}\|_{L^\infty} \|\nabla \rho\|_{L^2}^2 \leq C \|\bar{u}\|_{H^3} \|\rho\|_{H^2}^2,
$$

$$
I_{31}^2 \leq \|\nabla^2 \bar{u}\|_{L^1} \|\nabla \rho\|_{L^2} \|\rho\|_{L^6} + \|\nabla \bar{u}\|_{L^\infty} \|\nabla \rho\|_{L^2} \leq C \|\bar{u}\|_{H^3} \|\rho\|_{H^2}^2.
$$

For the second derivative, it follows from (13)$_2$ that

$$
\frac{1}{2} \frac{d}{dt} \|\nabla^2 \rho\|_{L^2}^2 = -\int_{\mathbb{T}^3} \nabla^2 (\bar{u} \cdot \nabla \rho) : \nabla^2 \rho dx - \int_{\mathbb{T}^3} \nabla^2 (\rho \nabla \cdot \bar{u}) : \nabla^2 \rho dx \\
:= I_{32} + I_{32}.
$$

We estimate $I_{32}$ as

$$
I_{32} \leq \|\nabla^2 \bar{u}\|_{L^2} \|\nabla \rho\|_{L^6} \|\nabla^2 \rho\|_{L^2} + 2 \int_{\mathbb{T}^3} |\nabla \bar{u}| |\nabla^2 \rho|^2 dx + \int_{\mathbb{T}^3} \bar{u} \cdot \nabla |\nabla^2 \rho|^2 dx \\
\leq C \|\nabla^2 \bar{u}\|_{H^1} \|\nabla \rho\|_{H^2}^2 + C \|\nabla \bar{u}\|_{L^\infty} \|\nabla^2 \rho\|_{L^2}^2 \\
\leq C \|\bar{u}\|_{H^3} \|\rho\|_{H^2}^2,
$$

$$
I_{32} \leq \|\nabla^2 \bar{u}\|_{L^1} \|\nabla \rho\|_{L^2} \|\nabla^2 \rho\|_{L^2} + 2 \int_{\mathbb{T}^3} |\nabla \bar{u}| |\nabla^2 \rho|^2 dx + \int_{\mathbb{T}^3} \bar{u} \cdot \nabla |\nabla^2 \rho|^2 dx \\
\leq C \|\nabla^2 \bar{u}\|_{H^1} \|\nabla \rho\|_{H^2}^2 + C \|\nabla \bar{u}\|_{L^\infty} \|\nabla^2 \rho\|_{L^2}^2 \\
\leq C \|\bar{u}\|_{H^3} \|\rho\|_{H^2}^2.
$$
Then, it follows from (13)

Thus, we have

Since

Thus, we have

For the estimate of

Similarly, we can also have

This again yields

(ii) For the positiveness of \( \rho \), we define the characteristic \( x(s) := x(s; t, x) \):

Then, it follows from (13) _2_ that

(iii) For the estimate of \( p \), we use the result of (i):

Then, we have

Since \( \nabla p = \gamma \rho^{\gamma - 1} \nabla \rho \), we have

Thus, we have

Thus, we have

This yields

For the estimate of \( \partial_t \rho \), we take \( L^2 \)-norm on both sides of (13) _2_ to have

Thus, we have

Similarly, we can also have

This again yields

\[
\| \nabla \rho \|_{L^\infty(0,T)} \leq C \| \bar{u} \|_{L^\infty(0,T)} \leq \frac{\varepsilon_7}{16} \leq \varepsilon_7^{1/6}.
\]

(17)
For the second derivative of $p$, we have
\[
\|\nabla^2 p\|^2_{L^2} = \gamma^2 (\gamma - 1)^2 C(\rho)^{2(\gamma - 2)} \|\nabla \rho\|^4_{L^4} + \gamma^2 \|\nabla \rho\|^2_{L^6} \|\nabla^2 \rho\|^2_{L^2}
\]
\[
\leq C \varepsilon_1^{3/4} + C \varepsilon_1^{7/8},
\]
where $C(\rho)$ is defined by (17):
\[
C(\rho) := \begin{cases} C \varepsilon_1^{1/2}, & 1 < \gamma \leq 2 \\ \|\rho\|_{L^\infty}, & \gamma > 2 \end{cases}
\]
Thus, we have from (18)-(20)
\[
\|p\|_{L^\infty(0,T;H^2)} \leq C \varepsilon_1^{3/8}.
\]
On the other hand, since $\partial_t p = \gamma \rho^{\gamma - 1} \partial_t \rho$, we have
\[
\|p_t\|_{L^\infty(0,T;H^1)} \leq C \|p_t\|_{L^\infty(0,T;H^1)} \leq C \varepsilon_1^{3/8}.
\]

**Lemma 3.4.** Suppose that the initial data $(f_0, \rho_0, u_0)$ and $\bar{u}$ satisfy
\begin{enumerate}[(i)]
\item $\|f_0\|_{W^{1,\infty}}, \|u_0\|_{H^2} < \varepsilon_1.$
\item $\|\rho_0\|_{H^2} < \varepsilon_1^{1/16}, \inf_{x \in \mathbb{T}^3} \rho_0(x) \geq \varepsilon_1^{1/2},$
\item $\sup_{0 \leq t \leq T} ||\bar{u}(t)||_{H^2}^2 + ||\bar{u}||_{L^2(0,T;H^3)}^2 + ||\bar{u}_t||_{L^2(0,T;H^1)}^2 \leq \varepsilon_1^{1/7},$
\end{enumerate}
where $\varepsilon$ is a constant given in Lemma 3.2. Then, there exists a unique solution $u$ to (13)$_3$ satisfying
\[
\sup_{0 \leq t \leq T} \left( ||u(t)||_{H^2}^2 + ||\sqrt{\rho}u_t(t)||_{L^2}^2 \right) + ||u||_{L^2(0,T;H^3)}^2 + ||u_t||_{L^2(0,T;H^1)}^2 \leq \varepsilon_1^{1/7}.
\]

**Proof.** By the positivity of $\rho$ in Lemma 3.3, the linear momentum equation (13)$_3$ can be written as a linear parabolic system
\[
uu + \bar{u} \cdot \nabla u + \rho^{-1} L u = -\rho^{-1} \nabla p - \rho^{-1} \int_{\mathbb{R}^3} (\bar{u} - \xi) f d\xi.
\]
The existence and uniqueness of the linear parabolic system (21) are well-known. We also refer the reader to [36]. For the regularity of $u$, we multiply (13)$_3$ by $u$ and integrate over $\mathbb{T}^3$; then, we have
\[
\frac{1}{2} \int_{\mathbb{T}^3} \rho |u|^2 dx + \int_{\mathbb{T}^3} (\mu |\nabla u|^2 + (\lambda + \mu) |\nabla \cdot u|^2) dx
\]
\[
= \frac{1}{2} \int_{\mathbb{T}^3} \rho t |u|^2 dx - \int_{\mathbb{T}^3} \rho(\bar{u} \cdot \nabla) u \cdot u dx - \int_{\mathbb{T}^3} \nabla p \cdot u dx + \int_{\mathbb{T}^3 \times P(t)} (\xi - \bar{u}) \cdot u f d\xi dx
\]
\[
= - \int_{\mathbb{T}^3} \nabla p \cdot u dx + \int_{\mathbb{T}^3 \times P(t)} (\xi - \bar{u}) \cdot u f d\xi dx
\]
\[
\leq \|p\|_{L^2} \|\nabla u\|_{L^2} + C \|f\|_{L^\infty(1 + ||\bar{u}||_{L^2})} \|u\|_{L^2}
\]
\[
\leq \|p\|_{L^2} \|\nabla u\|_{L^2} + C \varepsilon_1^{2/3} \|u\|_{L^4},
\]
where we used Lemma 3.2, 3.3 and
\[
\frac{1}{2} \int_{\mathbb{T}^3} \rho_t |u|^2 dx = -\frac{1}{2} \int_{\mathbb{T}^3} \nabla \cdot (\rho \bar{u}) |u|^2 dx = \int_{\mathbb{T}^3} \rho \bar{u} \cdot \nabla u \cdot u dx.
\]
Then, by Young’s inequality, we have
\[
\frac{d}{dt}\left\| \sqrt{\rho} u \right\|_{L^2}^2 + 2\mu \|\nabla u\|_{L^2}^2 \leq \frac{C}{\mu} \varepsilon^{3/4} + \mu \|\nabla u\|_{L^2}^2 + C \varepsilon^{2/3} + C \varepsilon^{2/3} \|u\|_{L^2}^2.
\]
We integrate the above relation with respect to \( t \) and use (17) to obtain
\[
C \varepsilon^{1/2} \|u(t)\|_{L^2}^2 + \mu \int_0^t \|\nabla u(s)\|_{L^2}^2 ds \leq C \varepsilon^{2/3} \int_0^t \|u(s)\|_{L^2}^2 ds + C (\varepsilon^{2/3} + \varepsilon^{3/4}) T + \|\sqrt{\rho_0} u_0\|_{L^2}^2.
\]
We use Gronwall’s lemma to get
\[
\|u\|_{L^2(0,T;L^2)}^2 \leq C (\varepsilon_1^{1/6} + \varepsilon_1^{1/4}) T + \varepsilon_1 \leq \varepsilon_1^{1/7}.
\]
This and (22) imply that
\[
\|\nabla u\|_{L^2(0,T;L^2)}^2 \leq C (\varepsilon_1^{1/6} + \varepsilon_1^{1/4}) T + \varepsilon_1 \leq \varepsilon_1^{1/7}.
\]
For more regularity, we multiply (13) by \( u_t \) and integrate over \( T^3 \), then, we have
\[
\int_{T^3} \rho |u_t|^2 dx + \frac{d}{dt} \int_{T^3} \left( \frac{\mu}{2} \|\nabla u\|^2 + \frac{\lambda + \mu}{2} |\nabla \cdot u|^2 - p \nabla \cdot u - \int_{R^3} (\xi - \bar{u}) \cdot u f d\xi \right) dx
\]
\[
= - \int_{T^3} \mu \cdot \nabla u \cdot u_t dx - \int_{T^3} p \nabla \cdot u dx + \int_{T^3 \times \mu} \partial_t ((\xi - \bar{u}) f) \cdot u d\xi dx
\]
\[
:= \sum_{k=1}^3 I_{53}^k.
\]
We use Lemmas 3.2, 3.3 and (23) to estimate \( I_{53}^k \):
\[
I_{53}^k \leq \frac{1}{2} \int_{T^3} \rho |u_t|^2 dx + \frac{1}{2} \|\rho\|_{L^\infty} \|\bar{u}\|_{L^\infty}^2 \int_{T^3} |\nabla u|^2 dx
\]
\[
\leq \frac{1}{2} \int_{T^3} \rho |u_t|^2 dx + \frac{1}{2} \varepsilon^{7/16} + \varepsilon^{1/7} \|\nabla u\|_{L^2}^2,
\]
\[
I_{53}^k \leq \|p\|_{L^2} \|\nabla u\|_{L^2} \leq \varepsilon_1^{3/8} \|\nabla u\|_{L^2} \leq \varepsilon_1^{1/16} \|\nabla u\|_{L^2}^2 + \varepsilon_1^{11/16},
\]
\[
I_{53}^k \leq \int_{T^3 \times \mu} (|\xi| f_{\xi} + |\bar{u}_t| |f| + |\bar{u}| |f_{\xi}|) |u| d\xi dx
\]
\[
\leq C \|f\|_{W^{1,\infty}} \left( \|u\|_{L^2} + \|\bar{u}_t\|_{L^2} \|u\|_{L^2} + \|\bar{u}\|_{L^2} \|u\|_{L^2} \right)
\]
\[
\leq C \varepsilon_1^{1/3} (1 + \|\bar{u}\|_{L^2}^2).
\]
We integrate (24) over \([0,t]\) to obtain
\[
\frac{1}{2} \int_0^t \|\sqrt{\rho} u_t\|_{L^2}^2 ds + \frac{\mu}{2} \|\nabla u\|_{L^2}^2
\]
\[
\leq C (\|u_0\|_{H^1}^2 + \|\rho_0\|_{L^2}^2 + \|f_0\|_{L^\infty}^2) + \varepsilon_1^{1/16} \int_0^t \|\nabla u\|_{L^2}^2 ds + \varepsilon_1^{2/3} T
\]
\[
+ C \varepsilon_1^{2/3} \|\bar{u}\|_{L^2(0,T;L^2)}^2 + \int_{T^3} (p \nabla \cdot u + \int_{R^3} (\xi - \bar{u}) \cdot u f d\xi) dx
\]
\[
\leq C \varepsilon_1^{7/8} + \varepsilon_1^{1/16} \int_0^t \|\nabla u\|_{L^2}^2 ds + \varepsilon_1^{2/3} T + C \varepsilon_1^{2/3} + C \varepsilon_1^{3/4} + \frac{\mu}{4} \|\nabla u\|_{L^2}^2 + C \varepsilon_1^{2/3}.
\]
This yields
\[
\frac{1}{2} \int_0^t \left\| \sqrt{\rho u_t} \right\|^2 dt + \frac{\mu}{4} \int_0^t \left\| \nabla u_t \right\|^2 dt \leq C \varepsilon^{2/3}(1 + T) + \varepsilon_1^{1/6} \int_0^t \left\| \nabla u_t \right\|^2 dt.
\] (25)

We now use Gronwall’s lemma to obtain
\[
\left\| \nabla u_t \right\|^2 \leq C \varepsilon_1^{2/3} (1 + T) (1 + C \varepsilon_1^{1/6} T \exp (C \varepsilon_1^{1/6} T)) \leq C \varepsilon_1^{2/3} (1 + T) \leq \varepsilon_1^{9/14}.
\] (26)

Moreover, it follows from (25) and (26) that
\[
C \varepsilon_1^{1/2} \left\| u_t \right\|^2_{L^2(0,T;L^2)} \leq \frac{1}{2} \int_0^t \left\| \sqrt{\rho u_t} \right\|^2 dt \leq C \varepsilon_1^{1/6+9/14} T + C \varepsilon_1^{2/3} (1 + T),
\]
which yields
\[
\left\| u_t \right\|^2_{L^2(0,T;L^2)} \leq C \varepsilon_1^{1/6+1/7} T + C \varepsilon_1^{1/6} (1 + T) < \varepsilon_1^{1/7}.
\] (27)

We next differentiate (13) with respect to \( t \) to obtain
\[
\rho u_{tt} + p_t u_t + L u_t = -\partial_t (\rho \bar{u} \cdot \nabla u) - \nabla p_t + \partial_t \int_{\mathbb{R}^3} (\xi - \bar{u}) f d\xi.
\]

We multiply this by \( \partial_t u \) and integrate it over \( \mathbb{T}^3 \) to obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} \rho |u_t|^2 dx + \int_{\mathbb{T}^3} (\mu |\nabla u_t|^2 + (\lambda + \mu) \nabla \cdot u_t|^2) dx
\]
\[
\quad = \frac{1}{2} \int_{\mathbb{T}^3} \rho_t |u_t|^2 dx - \int_{\mathbb{T}^3} \partial_t (\rho \bar{u} \cdot \nabla u) \cdot u_t dx - \int_{\mathbb{T}^3} \nabla p_t \cdot u_t dx
\]
\[
\quad + \int_{\mathbb{T}^3 \times P(t)} \partial_t \left( (\xi - \bar{u}) f \right) \cdot u_t d\xi dx
\]
\[
\quad = - \int_{\mathbb{T}^3} \partial_t (\rho \bar{u} \cdot \nabla u) \cdot u_t dx \quad \int_{\mathbb{T}^3} \nabla p_t \cdot u_t dx + \int_{\mathbb{T}^3 \times P(t)} \partial_t \left( (\xi - \bar{u}) f \right) \cdot u_t d\xi dx
\]
\[
\quad =: \sum_{k=1}^{3} \mathcal{I}_{34}^k.
\]

We use Lemma 3.2, 3.3 and (26) to estimate \( \mathcal{I}_{34}^k \) as follows.
\[
\mathcal{I}_{34}^k \leq \| \bar{u} \|_{L^\infty} \| |u_t| \|_{L^2} \| \nabla u_t \|_{L^2} \| u_t \|_{L^6} + \| \rho \|_{L^\infty} \| \bar{u}_t \|_{L^2} \| \nabla u_t \|_{L^2} \| u_t \|_{L^6}
\]
\[
\quad \leq C \varepsilon_1^{1/14+7/16} (1 + \varepsilon_1^{1/14}) \| \bar{u}_t \|_{H^2} \| u_t \|_{H^2}
\]
\[
\quad \leq C \varepsilon_1^{7/16} (\| \bar{u}_t \|_{H^2} + \| u_t \|_{H^2} + 1),
\]
\[
\mathcal{I}_{34}^k \int_{\mathbb{T}^3} \int_{P(t)} |f_t| dx \quad \leq \| |f_t| \|_{L^2} \| u_t \|_{L^2} \leq C \varepsilon_1^{3/8} \| u_t \|_{L^2} \leq C \varepsilon_1^{3/8} (\| \nabla u_t \|_{L^2}^2 + 1),
\]
\[
\mathcal{I}_{34}^k \int_{\mathbb{T}^3 \times P(t)} (|f_t| \| u_t \|_{L^2} + \| u_t \|_{H^2}) dx
\]
\[
\quad \leq C \varepsilon_1^{1/4} (1 + \| \bar{u} \|_{L^2}) \| u_t \|_{L^2}
\]
\[
\quad \leq C \varepsilon_1^{3/4} (\| u_t \|_{L^2}^2 + \| \bar{u}_t \|_{L^2} + 1).
\]

Then, we use the smallness of \( \varepsilon_1 \) to get
\[
\frac{d}{dt} \int_{\mathbb{T}^3} \rho |u_t|^2 dx + \mu \int_{\mathbb{T}^3} |\nabla u_t|^2 dx \leq C \varepsilon_1^{3/8} (\| u_t \|_{L^2}^2 + \| \bar{u} \|_{H^2} + 1).
\] (28)
We integrate (28) over $[τ, t]$ with $0 < τ < t$ to find

$$
\|\sqrt{\rho u_t}\|_{L^2}^2 + \mu \int_τ^t \|\nabla u_t(s)\|_{L^2}^2 ds
\leq \|\sqrt{\rho u_τ}\|_{L^2}^2 + Cε_1^{3/8} \int_τ^t \|u_t\|_{L^2}^2 ds + Cε_1^{3/8} T
$$

(29)

$$
\leq \|\sqrt{\rho u_τ}\|_{L^2}^2 + Cε_1^{3/8} + Cε_1^{3/8} T,
$$

where we used (27).

Finally, we estimate $\|\sqrt{\rho u_t(τ)}\|_{L^2}^2$ as follows. It follows from (13) that

$$
ρ^2|u_t|^2 \leq C(ρ^2|\bar{u}|^2|∇u|^2 + |Lu + ∇p|^2 + \int_{P(t)} (|ξ|^2 + |\bar{u}|^2)f^2 dξ).
$$

We integrate this over $T^3$ to get

$$
\int_{T^3} ρ|u_t|^2 dx \leq C \int_{T^3} (ρ|\bar{u}|^2|∇u|^2 + ρ^{-1}|Lu + ∇p|^2 + ρ^{-1}\int_{P(t)} (|ξ|^2 + |\bar{u}|^2)f^2 dξ) dx.
$$

(30)

We use the same estimates as (18)-(19) to get

$$
\|ρ_0\|_{L^1}^2 \leq C\|ρ_0\|_{L^∞}^{2(γ-1)}\|ρ_0\|_{H^1}^2 \leq Cε_1^{7/8}.
$$

This yields

$$
\int_{T^3} ρ_0|u_0|^2|∇u_0|^2 dx \leq \|ρ_0\|_{L^∞} \|u_0\|_{L^∞}^2 \|∇u_0\|_{L^2}^2 \leq ε_1^2,
$$

$$
\int_{T^3} ρ_0^{-1}|Lu_0 + ∇ρ_0|^2 dx \leq Cε_1^{-1/2}(\|u_0\|_{H^2}^2 + \|ρ_0\|_{H^1}^2) \leq ε_1^{3/8},
$$

$$
\int_{T^3 \times P(t)} ρ_0^{-1}(|ξ|^2 + |u_0|^2)f_0^2 dξ dx \leq Cε_1^{-1/2}(\|u_0\|_{L^2}^2 + 1)\|f_0\|_{L^∞}^2 \leq ε_1.
$$

Then, it follows from (30) that

$$
\limsup_{τ \to 0} \|\sqrt{\rho u_t(τ)}\|_{L^2}^2 \leq Cε_1^{3/8}.
$$

Thus, we take $τ \to 0$ in (29) and use (17) to obtain

$$
\|\sqrt{\rho u_t}\|_{L^2}^2 + \mu \int_0^t \|\nabla u_t(s)\|_{L^2}^2 ds \leq Cε_1^{3/8} (1 + T) < ε_1^{1/7}.
$$

which yields

$$
\|\sqrt{ρ u_t}\|_{L^∞(0,T;L^2)} + \|∇u_t(s)\|_{L^2(0,T;L^2)}^2 < Cε_1^{1/7},
$$

(31)

To obtain further regularity, we consider the elliptic system:

$$
Lu = F, \quad F = -ρu_t - ρ\bar{u} \cdot ∇u - ∇p + \int_{P(t)} (ξ - \bar{u}) f dξ.
$$

(32)

We take $L^2$-norm on (32) and use the Calderon-Zygmund inequality to obtain

$$
\|∇^2 u\|_{L^2} \leq C\|Lu\|_{L^2} = C|μ - ρu_t - ρ\bar{u} \cdot ∇u - ∇p + \int_{P(t)} (ξ - \bar{u}) f dξ|_{L^2}
$$

$$
\leq C(\|\sqrt{ρ}\|_{L^∞}\|\sqrt{ρ u_t}\|_{L^2} + \|ρ\bar{u}\|_{L^∞}\|∇u\|_{L^2} + \|∇p\|_{L^2} + (\|\bar{u}\|_{L^2} + 1)\|f\|_{L^∞})
$$

$$
\leq Cε_1^{7/32}\|\sqrt{ρ u_t}\|_{L^∞(0,T;L^2)} + Cε_1^{3/8} + Cε_1^{2/3}.
$$
By (31), we have
\[ \|\nabla^2 u\|_{L^\infty(0,T;L^2)} \leq \varepsilon_1^{1/7}. \] (33)
Furthermore, it follows from Lemma 3.3 and (26) and (33) that
\[ \|\nabla(p\bar{u} \cdot \nabla u)\|_{L^2}^2 \leq C(\|\bar{u}\|_{L^\infty}^2 \|\nabla \bar{p}\|_{L^2}^2 \|\nabla u\|_{L^2}^2 + \|\bar{p}\|_{L^\infty}^2 \|\nabla \bar{u}\|_{L^2}^2 \|\nabla u\|_{L^2}^2 + \|\bar{p}\|_{L^\infty}^2 \|\nabla \bar{u}\|_{L^2}^2)
   \leq \varepsilon_1^{2/7+7/8}. \]
This implies
\[ \|\nabla^3 u\|_{L^2}^2 \leq C\|\nabla Lu\|_{L^2}
   \leq C\|\nabla(p\bar{u} + \rho\bar{u} \cdot \nabla p - \int_{p(t)}^\xi (\xi - \bar{u})f \, d\xi)\|_{L^2}
   \leq C(\|\rho\|_{L^\infty}^2 \|\nabla u_t\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 \|u_t\|_{L^2}^2 + \|\nabla(p\bar{u} \cdot \nabla u)\|_{L^2}^2
   \leq C\varepsilon_1^{3/4}(\|\nabla u_t\|_{L^2}^2 + \|u_t\|_{L^2}^2 + 1)\]
Thus, we have
\[ \|\nabla^3 u\|_{L^2(0,T;L^2)} \leq C\varepsilon_1^{3/4}\left(\|\nabla u_t\|_{L^2(0,T;L^2)}^2 + \|u_t\|_{L^2(0,T;L^2)}^2 + T\right) < \varepsilon_1^{1/7}. \]

3.2. Proof of Theorem 2.3. Since the proof is rather lengthy, we split its presentation into several lemmas. The existence of the desired strong solutions will be addressed using Schauder’s fixed-point theorem, which does not guarantee the uniqueness itself. Thus, we first study the uniqueness issue in the following lemma.

**Lemma 3.5.** (Uniqueness) Let \( [f, \rho, u] \) and \( [\bar{f}, \bar{\rho}, \bar{u}] \) be the two strong solutions to the system (1) - (3) corresponding to the same initial data \( (f_0, \rho_0, u_0) \). Then, we have
\[ f \equiv \bar{f} \quad \text{in } W^{1,\infty}(\mathbb{T}^3 \times \mathbb{R}^3 \times [0,T]), \]
\[ \rho \equiv \bar{\rho} \quad \text{in } L^\infty(0,T;H^3(\mathbb{T}^3)), \]
\[ u \equiv \bar{u} \quad \text{in } L^\infty(0,T;H^2(\mathbb{T}^3)) \cap L^2(0,T;H^3(\mathbb{T}^3)). \]

**Proof.** We set
\[ \Delta_d(t) := \|f - \bar{f}\|_{L^\infty}^2 + \|\rho - \bar{\rho}\|_{H^3}^2 + \|(p - \bar{p})(t)\|_{H^1}^2 + \|(u - \bar{u})(t)\|_{H^1}^2. \]
Then, by the same arguments as in Lemma 2, \( \Delta_d(t) \) satisfies the inequality
\[ \frac{d}{dt} \Delta_d + \|u - \bar{u}\|_{H^2}^2 \leq C(\|u\|_{H^3} + \|\bar{u}\|_{H^3} + \|u_t\|_{H^1} + 1)\Delta_d, \quad \Delta_d(0) = 0. \]
Then, the standard Gronwall’s lemma implies
\[ \Delta_d(t) = 0, \quad \text{i.e., } f \equiv \bar{f} \quad \text{in } L^\infty(\mathbb{T}^3 \times \mathbb{R}^3 \times [0,T]), \]
\[ \rho \equiv \bar{\rho} \quad \text{in } L^\infty(0,T;H^1(\mathbb{T}^3)), \]
\[ u \equiv \bar{u} \quad \text{in } L^\infty(0,T;H^2(\mathbb{T}^3)) \cap L^2(0,T;H^3(\mathbb{T}^3)) \cap H^1(0,T;H^1(\mathbb{T}^3)). \]
This easily implies the uniqueness of strong solutions satisfying the desired regularity. \( \square \)
We next prepare the setting for Schauder’s fixed-point theorem. For this, we need to introduce compact and convex set $\mathcal{X}$ in $L^2(0,T;H^2(\mathbb{T}^3))$ and continuous mapping on $\mathcal{X}$.

### 3.2.1. Setup for Schauder’s fixed-point theorem

Define a set $\mathcal{X}$:

$$
\mathcal{X} := \{ u \mid \| u \|_{L^\infty(0,T;H^2)} + \| u_t \|_{L^2(0,T;H^3)} + \| u_{tt} \|_{L^2(0,T;H^1)} \leq \varepsilon^{1/7} \}.
$$

It should be noted that $\mathcal{X}$ is clearly convex, and compact in $L^2(0,T;H^2(\mathbb{T}^3))$ by the Aubin-Lions compactness theorem. Indeed, by the compactness theorem,

$$
\{ u \in L^2(0,T;H^3(\mathbb{T}^3)) \mid u_t \in L^2(0,T;H^1(\mathbb{T}^3)) \}
$$

is compactly embedded in $L^2(0,T;H^2(\mathbb{T}^3))$.

Thus, $Y := \{ u \mid \| u \|_{L^2(0,T;H^3)} + \| u_t \|_{L^2(0,T;H^1)} \leq \varepsilon^{1/7} \}$ is compact in $L^2(0,T;H^2(\mathbb{T}^3))$. Since $\mathcal{X}$ is a closed subset of $Y$, $\mathcal{X}$ is also compact in $L^2(0,T;H^2(\mathbb{T}^3))$.

We are now ready to define a nonlinear map $\Phi$ from $\mathcal{X}$ to $\mathcal{X}$ as follows.

$$
\bar{u} \in \mathcal{X} \Rightarrow u = \Phi(\bar{u}) \text{ is the solution to (13).} \tag{34}
$$

Then, due to Lemma 3.4, the above map $\Phi$ is well-defined. Here, the topology of $\mathcal{X}$ is inherited from that of $L^2(0,T;H^2(\mathbb{T}^3))$. Finally, once we have proved the continuity of $\Phi$, we can complete the proof of the existence by the standard fixed-point theorem of Schauder.

### 3.2.2. Continuity of the map $\Phi$

Let $\bar{u} \in \mathcal{X}$ and $\{ \bar{u}_n \}$ be any sequence in $\mathcal{X}$ such that

$$
\| \bar{u}_n - \bar{u} \|_{L^2(0,T;H^2)} \rightarrow 0, \quad \text{as } n \rightarrow \infty.
$$

On the other hand, by Sobolev inequality, this implies

$$
\| \bar{u}_n - \bar{u} \|_{L^2(0,T;L^\infty)} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \tag{35}
$$

To establish the continuity of $\Phi$, it suffices to show that

$$
\| u_n - u \|_{L^2(0,T;H^2)} \rightarrow 0, \quad \text{as } n \rightarrow \infty,
$$

where $u_n = \Phi(\bar{u}_n)$ and $u = \Phi(\bar{u})$. For this, we need to show that as $n \rightarrow \infty$,

$$
\| f_n - f \|_{L^\infty} + \| \rho_n - \rho \|_{L^\infty(0,T;H^1)} + \| p_n - p \|_{L^\infty(0,T;H^1)} \rightarrow 0, \tag{36}
$$

where $f_n, \rho_n$, and $p_n$ are solutions of (13) corresponding to $\bar{u}_n$. In the following series of lemmas, we provide the estimate (71).

**Lemma 3.6.** Suppose that the relation (35) holds. Then, we have

$$
\lim_{n \rightarrow \infty} \| f_n - f \|_{L^\infty} = 0.
$$

**Proof.** It follows from the systems (13) corresponding to $\bar{u}_n$ and $\bar{u}$ that we have

$$
\begin{align*}
\partial_t (f_n - f) + \xi \cdot \nabla (f_n - f) + (F_n(f_n) + \bar{u}_n - \xi) \cdot \nabla \xi (f_n - f) \\
= -\nabla \xi f \cdot (F_n(f_n) - F_n(f) + \bar{u}_n - \bar{u}) - (f_n - f) \nabla \xi \cdot F_n(f_n) \\
- f \nabla \xi \cdot (F_n(f_n) - F_n(f)) + 3(f_n - f).
\end{align*}
$$

(37)
Lemma 3.7. Suppose that the relation (37) along the particle trajectory \((x_n(t), \xi_n(t))\) associated with \(\bar{u}_n\) to obtain

\[
(f_n - f)(x_n(t), \xi_n(t), t)
\]

\[
= -\int_0^t \left[ \nabla f \cdot ( F_a(f_n) - F_a(f) + \bar{u}_n - \bar{u}) + (f_n - f) \nabla \xi \cdot F_a(f_n) \right] (x_n(s), \xi_n(s), s) ds
\]

\[
- \int_0^t \left[ f \nabla \xi \cdot ( F_a(f_n) - F_a(f) ) - 3(f_n - f) \right] (x_n(s), \xi_n(s), s) ds.
\]

We now claim:

(i) \( \|f\|_{W^{1, \infty}} \leq C \), \( \| \nabla \xi \cdot F_a(f_n) \|_{L^\infty} \leq C \),

(ii) \( \|F_a(f_n) - F_a(f)\|(t, x_n(t), \xi_n(t)) \leq C \|f_n - f\|(t) \|_{L^\infty} \),

(iii) \( \| \nabla \xi \cdot ( F_a(f_n) - F_a(f) )\|(t, x_n(t), \xi_n(t)) \leq C \|f_n - f\|(t) \|_{L^\infty} \),

where \( C \) is independent of \( n \).

Proof of claim. First, (38) is easily obtained such as Lemma A.4 in [1]. Next, since \( \|\bar{u}_n\|_{L^\infty(0,T; L^\infty)} \leq K \), we can have the same estimate \( \sup_{0 \leq t \leq T} |\xi_n(t)| \leq C \) as Lemma 3.1 where \( C \) is independent of \( n \). This implies (38).

Hence,

\[
\|f_n - f\|_{L^\infty} \leq C \|\bar{u}_n - \bar{u}\|_{L^2(0,T; L^\infty)},
\]

In (37), we use (38) to find

\[
\| (f_n - f)(t) \|_{L^\infty} \leq C \|\bar{u}_n - \bar{u}\|_{L^2(0,T; L^\infty)} + C \int_0^t \| (f_n - f)(s) \|_{L^\infty} ds.
\]

Then, Gronwall’s lemma implies

\[
\| (f_n - f)(t) \|_{L^\infty} \leq \|\bar{u}_n - \bar{u}\|_{L^2(0,T; L^\infty)} e^{CT} \to 0, \quad \text{as } n \to \infty.
\]

\[
\square
\]

Lemma 3.7. Suppose that the relation (35) holds. Then, we have

\[
\lim_{n \to \infty} \|\rho_n - \rho\|_{L^\infty(0,T; H^1)} = 0.
\]

Proof. We subtract (13) from itself corresponding to \(\bar{u}_n\) to obtain

\[
\partial_t (\rho_n - \rho) + (\bar{u}_n - \bar{u}) \cdot \nabla \rho_n + \bar{u} \cdot \nabla (\rho_n - \rho)
\]

\[
+ (\rho_n - \rho) \nabla \cdot \bar{u}_n + \rho \nabla \cdot (\bar{u}_n - \bar{u}) = 0.
\]

(39)

• (Estimate of \( \|\rho_n - \rho\|_{L^\infty(0,T; L^2)} \)):

We multiply this equation by \( \rho_n - \rho \) and integrate the resulting relation over \( T^3 \) to obtain

\[
\frac{1}{2} \frac{d}{dt} \|\rho_n - \rho\|(t)^2_{L^2} \leq \int_{T^3} (\bar{u}_n - \bar{u}) \cdot \nabla \rho_n \cdot (\rho_n - \rho) dx + \frac{1}{2} \int_{T^3} \bar{u} \cdot \nabla |\rho_n - \rho|^2 dx
\]

\[
+ \int_{T^3} \rho \nabla \cdot (\bar{u}_n - \bar{u}) \cdot (\rho_n - \rho) dx
\]

\[
:= \sum_{k=1}^{4} T^k_{35}.
\]

(40)
The terms $I_{35}^k$ can be estimated as
\begin{align*}
I_{35}^1 &\leq \|ho_n - \rho\|_{L^2} \|
abla \rho_n\|_{L^2} \|ar{u}_n - \bar{u}\|_{L^6} \leq C \|ho_n - \rho\|_{L^2} \|ar{u}_n - \bar{u}\|_{H^1}, \\
I_{35}^2 &=- \frac{1}{2} \int_{\mathbb{T}^3} \rho_n - \rho \right)^2 \nabla \cdot \bar{u} dx \leq \|
abla \bar{u}\|_{L^\infty} \|ho_n - \rho\|^2_{L^2}, \\
I_{35}^3 &\leq \|
abla \bar{u}_n\|_{L^\infty} \|ho_n - \rho\|^2_{L^2}, \\
I_{35}^4 &\leq \|ho\|_{L^\infty} \|
abla (\bar{u}_n - \bar{u})\|_{L^2} \|ho_n - \rho\|_{L^2} \leq C \|ar{u}_n - \bar{u}\|_{H^1} \|ho_n - \rho\|_{L^2}.
\end{align*}

Then, we use Young’s inequality to have
\begin{equation}
\frac{d}{dt} \|ho_n - \rho\|^2_{L^2} \leq C(\|ar{u}\|_{H^3} + \|ar{u}_n\|_{H^3} + 1) \|ho_n - \rho\|^2_{L^2} + C \|ar{u}_n - \bar{u}\|^2_{H^1}. \quad (41)
\end{equation}

- (Estimate of $\|
abla (\rho_n - \rho)\|_{L^\infty(0,T;L^2)}$):

We take the gradient of (49) and multiply the resulting relation by $\nabla (\rho_n - \rho)$, and then integrate it over $\mathbb{T}^3$ to obtain
\begin{equation}
\frac{1}{2} \frac{d}{dt} \|
abla (\rho_n - \rho)(t)\|^2_{L^2} \leq - \int_{\mathbb{T}^3} \left[ \nabla (\bar{u}_n - \bar{u}) \cdot \nabla \rho_n + (\bar{u}_n - \bar{u}) \cdot \nabla^2 \rho_n \\
+ \nabla \bar{u} \cdot \nabla (\rho_n - \rho) + \bar{u} \cdot \nabla^2 (\rho_n - \rho) \\
+ \nabla (\rho_n - \rho) \cdot \nabla \cdot \bar{u}_n + (\rho_n - \rho) \nabla \cdot (\nabla \cdot \bar{u}_n) \\
+ \nabla \rho \nabla \cdot (\bar{u}_n - \bar{u}) + \rho \nabla (\nabla \cdot (\bar{u}_n - \bar{u})) \right] \cdot \nabla (\rho_n - \rho) dx \leq \sum_{k=1}^{8} I_{36}^k.
\end{equation}

The terms $I_{36}^k$ can be estimated as
\begin{align*}
I_{36}^1 &\leq \|
abla (\bar{u}_n - \bar{u})\|_{L^1} \|
abla \rho_n\|_{L^6} \|
abla (\rho_n - \rho)\|_{L^2} \leq C \|ar{u}_n - \bar{u}\|_{H^2} \|
abla (\rho_n - \rho)\|_{L^2}, \\
I_{36}^2 &\leq \|ar{u}_n - \bar{u}\|_{L^\infty} \|
abla^2 \rho_n\|_{L^2} \|
abla (\rho_n - \rho)\|_{L^2} \leq C \|ar{u}_n - \bar{u}\|_{H^2} \|
abla (\rho_n - \rho)\|_{L^2}, \\
I_{36}^3 &\leq \|
abla \bar{u}\|_{L^\infty} \|
abla (\rho_n - \rho)\|^2_{L^2}, \\
I_{36}^4 &=- \frac{1}{2} \int_{\mathbb{T}^3} \nabla \cdot \bar{u} \|
abla (\rho_n - \rho)\|^2 dx \leq \frac{1}{2} \|
abla \bar{u}\|_{L^\infty} \|
abla (\rho_n - \rho)\|^2_{L^2}, \\
I_{36}^5 &\leq \|
abla \bar{u}_n\|_{L^\infty} \|
abla (\rho_n - \rho)\|^2_{L^2}, \\
I_{36}^6 &\leq \|ho_n - \rho\|_{L^3} \|
abla^2 \bar{u}_n\|_{L^6} \|
abla (\rho_n - \rho)\|_{L^2} \leq C \|ar{u}_n\|_{H^3} \|ho_n - \rho\|^2_{H^1}, \\
I_{36}^7 &\leq \|
abla (\bar{u}_n - \bar{u})\|_{L^1} \|
abla \rho\|_{L^6} \|
abla (\rho_n - \rho)\|_{L^2} \leq C \|ar{u}_n - \bar{u}\|_{H^2} \|
abla (\rho_n - \rho)\|_{L^2}, \\
I_{36}^8 &\leq \|ho\|_{L^\infty} \|
abla^2 (\bar{u}_n - \bar{u})\|_{L^2} \|
abla (\rho_n - \rho)\|_{L^2} \leq C \|ar{u}_n - \bar{u}\|_{H^2} \|
abla (\rho_n - \rho)\|_{L^2}.
\end{align*}

Then, we use Young’s inequality to have
\begin{equation}
\frac{d}{dt} \|
abla (\rho_n - \rho)\|^2_{L^2} \leq C(\|ar{u}\|_{H^3} + \|ar{u}_n\|_{H^3} + 1) \|ho_n - \rho\|^2_{H^1} + C \|ar{u}_n - \bar{u}\|^2_{H^2}. \quad (43)
\end{equation}

We combine (41) with (43) and Gronwall’s lemma to obtain
\begin{align*}
\|ho_n - \rho\|_{L^\infty(0,T;H^1)} &\leq C \|ar{u}_n - \bar{u}\|_{L^2(0,T;H^2)} \exp(C(\|ar{u}\|_{L^1(0,T;H^3)} + \|ar{u}_n\|_{L^1(0,T;H^3)} + 1)) \\
&\leq C \|ar{u}_n - \bar{u}\|_{L^2(0,T;H^2)} \to 0.
\end{align*}
Lemma 3.8. Suppose that the relation (35) holds. Then, we have
\[ \lim_{n \to \infty} \|p_n - p\|_{L^\infty(0,T;H^1)} = 0. \]

Proof. It follows from (13) that
\[ \partial_t (p_n - p) + (\bar{u}_n - \bar{u}) \cdot \nabla p + \bar{u} \cdot \nabla (p_n - p) + \gamma (p_n - p) \nabla \cdot (\bar{u}_n - \bar{u}) = 0. \] (44)

We multiply this equation by \( p_n - p \) and integrate it over \( \mathbb{T}^3 \) to find
\[ \frac{d}{dt} \|p_n - p\|_{L^2}^2 \leq C(\|\bar{u}\|_{L^3} + 1)\|p_n - p\|_{L^2}^2 + C\|\bar{u}_n - \bar{u}\|_{H^1}^2. \] (45)

which we estimated in the same way as (40)-(41).

On the other hand, for the estimate of \( \|\nabla (p_n - p)\|_{L^\infty(0,T;L^2)} \), we take the gradient of (44) and multiply the resulting relation by \( \nabla (p_n - p) \), and then integrate it over \( \mathbb{T}^3 \) to obtain
\[ \frac{d}{dt} \|\nabla (p_n - p)\|_{L^2}^2 \leq C(\|\bar{u}\|_{L^3} + 1)\|p_n - p\|_{L^2}^2 + C\|\bar{u}_n - \bar{u}\|_{H^2}^2. \] (46)

which we estimated in the same way as (42)-(43).

Finally, we combine (45) with (46) and Gronwall’s lemma to get
\[ \|p_n - p\|_{L^\infty(0,T;H^1)} \leq C\|\bar{u}_n - \bar{u}\|_{L^2(0,T;H^2)} \exp(C(\|\bar{u}\|_{L^1(0,T;H^3)} + 1)) \leq C\|\bar{u}_n - \bar{u}\|_{L^2(0,T;H^2)} \to 0. \] (47)

We are now ready to present a proof of the continuity of \( \Phi \).

Proposition 2. The operator \( \Phi : \mathcal{X} \to \mathcal{X} \) defined by (34) is continuous.

Proof. We need to verify
\[ \|\Phi(\bar{u}_n) - \Phi(u)\|_{L^2(0,T;H^2)} \leq \|u_n - u\|_{L^2(0,T;H^2)} \to 0, \quad \text{as} \quad n \to \infty. \]

For this, we subtract (13) from itself corresponding to \( \bar{u}_n \), and then we have
\[ \rho \partial_t (u_n - u) + (\rho_n - \rho) \partial_t u + \rho_n (\bar{u}_n \cdot \nabla)(u_n - u) + \rho_n (\bar{u}_n - \bar{u}) \cdot \nabla u + (\rho_n - \rho) (\bar{u} \cdot \nabla) u + L(u_n - u) + \nabla (p_n - p) = \int_{\mathbb{R}^3} \xi (f_n - f) - \bar{u}_n (f_n - f) + (\bar{u} - \bar{u}_n) f d\xi. \] (48)

We multiply this equation by \( u_n - u \) and integrate it over \( \mathbb{T}^3 \) to obtain
\[ \frac{1}{2} \frac{d}{dt} \|\sqrt{\rho}(u_n - u)(t)\|_{L^2}^2 + \mu \|\nabla (u_n - u)(t)\|_{L^2}^2 \leq \frac{1}{2} \|\rho_t\|_{L^\infty} \|u_n - u\|_{L^2} \|u_n - u\|_{L^2} + \|\partial_t u\|_{L^3} \|u_n - u\|_{L^6} \|\rho_n - \rho\|_{L^2} + \|\rho_n \bar{u} - \nabla (u_n - u)\|_{L^2} \|u_n - u\|_{L^2} + \|\rho_n \bar{u}_n - \bar{u}_n \|_{L^6} \|u_n - u\|_{L^6} + \|\bar{u}_n - \bar{u}\|_{L^6} \|u_n - u\|_{L^6} + \|\rho_n - \rho\|_{L^2} \|u_n - u\|_{L^2} + \|p_n - p\|_{L^2} \|\nabla (u_n - u)\|_{L^2} + C(\|f_n - f\|_{L^\infty} + \|\bar{u}_n - \bar{u}\|_{L^2}) \|u_n - u\|_{L^2}. \]
We use Young’s inequality to have
\[
\frac{d}{dt} \| \sqrt{\rho(u_n - u)(t)} \|_{L^2}^2 + C \| \nabla (u_n - u)(t) \|_{L^2}^2 \leq C \left( \| u_n - u \|_{L^2}^2 \\
+ (\| \partial_t u_n \|_{H^1}^2 + 1) \| \rho_n - \rho \|_{L^2}^2 + \| \bar{u}_n - \bar{u} \|_{H^1}^2 + \| p_n - p \|_{L^2}^2 + \| f_n - f \|_{L^\infty}^2 \right).
\]
We integrate this in time and use the strict positiveness of \( \rho \) to obtain
\[
\| (u_n - u)(t) \|_{L^2}^2 \leq C \int_0^t \| (u_n - u)(s) \|_{L^2}^2 ds + C A_n
\]
where
\[
A_n := \| \bar{u}_n - \bar{u} \|_{L^2(0,T;H^1)}^2 + \| \rho_n - \rho \|_{L^\infty(0,T;H^1)}^2 + \| p_n - p \|_{L^\infty(0,T;H^1)}^2 + \| f_n - f \|_{L^\infty}^2.
\]
Therefore, we use Lemmas 3.9 - 3.11 to get
\[
\sup_{0 \leq t \leq T} \| (u_n - u)(t) \|_{L^2(T^3)}^2 \leq A_n \to 0. \tag{49}
\]
For the estimate of \( \| \partial_t (u_n - u) \|_{L^2(0,T;L^2)} \), we multiply (75) by \( \partial_t (u_n - u) \) and integrate over \( T^3 \) to find
\[
\| \sqrt{\rho} \partial_t (u_n - u) \|_{L^2}^2 + \frac{\mu}{2} \frac{d}{dt} \| \nabla (u_n - u) \|_{L^2}^2 \leq \left( \| \partial_t u_n \|_{L^2} \| \rho_n - \bar{\rho} \|_{L^4} + \| \rho_n \bar{u}_n \|_{L^\infty} \| \nabla (u_n - u) \|_{L^2} + \| \rho_n \|_{L^\infty} \| \nabla u \|_{L^3} \| \bar{u}_n - \bar{\bar{u}} \|_{L^6} \\
+ \| \bar{\bar{u}} \|_{L^\infty} \| \nabla u \|_{L^3} \| \rho_n - \bar{\rho} \|_{L^6} + \| \nabla (\rho_n - p) \|_{L^2} + C \| f_n - f \|_{L^\infty} + C \| \bar{u}_n - \bar{\bar{u}} \|_{L^2} \right) \times \| \partial_t (u_n - u) \|_{L^2}.
\]
We use the positiveness of \( \rho \), and then Young’s inequality to have
\[
\| \partial_t (u_n - u) \|_{L^2}^2 + \frac{d}{dt} \| \nabla (u_n - u) \|_{L^2}^2 \leq C \left( \| \nabla (u_n - u) \|_{L^2}^2 \\
+ (\| \partial_t u_n \|_{H^1}^2 + 1) \| \rho_n - \rho \|_{H^1}^2 + \| \bar{u}_n - \bar{\bar{u}} \|_{H^1}^2 + \| p_n - p \|_{H^1}^2 + \| f_n - f \|_{L^\infty}^2 \right).
\]
By Gronwall’s lemma, we have
\[
\| \partial_t (u_n - u) \|_{L^2(0,T;L^2)}^2 + \| \nabla (u_n - u) \|_{L^\infty(0,T;L^2(T^3))}^2 \leq A_n \to 0. \tag{50}
\]
Finally, for the estimate of \( \| \nabla^2 (u_n - u) \|_{L^2(0,T;L^2) \to T^3} \), we have
\[
\| \nabla^2 (u_n - u) \|_{L^2} \leq C \| L(u_n - u) \|_{L^2} \\
\leq \left\| - \rho \partial_t (u_n - u) - (\rho_n - \rho) \partial_t u_n - \rho_n (\bar{u}_n \cdot \nabla)(u_n - u) - \rho_n (\bar{\bar{u}}_n - \bar{\bar{u}}) \cdot \nabla u \\
- (\rho_n - \rho) (\bar{\bar{u}} \cdot \nabla) u - \nabla (p_n - p) \right\|_{L^2} + \left\| \int_{\mathbb{R}^3} \xi (f_n - f) - \bar{u}_n (f_n - f) + (\bar{u}_n - \bar{\bar{u}}_n) f \right\|_{L^2} \\
\leq \| \rho \|_{L^\infty} \| \partial_t (u_n - u) \|_{L^2} + \| \rho_n - \rho \|_{L^4} \| \partial_t u_n \|_{L^4} + \| \rho_n \bar{u}_n \|_{L^\infty} \| \nabla (u_n - u) \|_{L^2} \\
+ \| \rho_n \|_{L^\infty} \| \bar{u}_n - \bar{\bar{u}} \|_{L^4} \| \nabla u \|_{L^4} + \| \bar{\bar{u}} \|_{L^\infty} \| \rho_n - \rho \|_{L^4} \| \nabla u \|_{L^4} + \| \nabla (p_n - p) \|_{L^2} \\
+ C \| f_n - f \|_{L^\infty} + C \| \bar{u}_n \|_{L^2} \| f_n - f \|_{L^\infty} + C \| f \|_{L^\infty} \| \bar{u}_n - \bar{\bar{u}}_n \|_{L^2}.
\]
Thus, we have
\[
\| \nabla^2 (u_n - u) \|_{L^2(0,T;L^2)} \leq C \left( \| \partial_t (u_n - u) \|_{L^2(0,T;L^2)} + \| \partial_t u_n \|_{L^2(0,T;H^1)} + \| \rho_n - \rho \|_{L^\infty(0,T;H^1)} \\
+ \| \nabla (u_n - u) \|_{L^2(0,T;L^2)} + \| \bar{u}_n - \bar{\bar{u}}_n \|_{L^2(0,T;H^1)} + \| p_n - p \|_{L^2(0,T;H^1)} + \| f_n - f \|_{L^\infty} \right).
\]
We now use Lemmas 3.9 - 3.11, (47) and (50) to get
\[ \| \nabla^2 (u_n - u) \|_{L^2(0,T;L^2)} \to 0, \quad \text{as } n \to \infty. \] (51)

Therefore, (49), (50), and (51) complete the proof.

4. **Asymptotic flocking dynamics.** In this section, we provide an asymptotic flocking estimate for the system (1) using the nonlinear Lyapunov functional approach.

4.1. **A Lyapunov functional.** In this part, we introduce a Lyapunov functional measuring the degree of flocking. Recall the desired Lyapunov functional \( \mathcal{L} \) introduced in Section 2.3.2:
\[ \mathcal{L}(t) := \mathcal{L}_p(t) + \mathcal{L}_f(t) + \mathcal{L}_d(t), \quad t \geq 0, \]
\[ \mathcal{L}_p(t) := \frac{1}{2} \int_{\mathbb{T}^3 \times \mathbb{R}^3} |\xi - \xi_c|^2 f d\xi dx, \quad \mathcal{L}_d(t) := |u_c - \xi_c|^2, \]
\[ \mathcal{L}_f(t) := \frac{1}{2} \int_{\mathbb{T}^3} \rho |u - u_c|^2 dx + \int_{\mathbb{T}^3} (\rho - \rho_c)^2 dx, \]

Here, \( \xi_c, u_c \) and \( \rho_c \) are defined in (12).

Note that the subfunctionals \( \mathcal{L}_p \) and \( \mathcal{L}_f \) measure the degree of flocking among the C-S particles and fluid themselves, respectively. In contrast, \( \mathcal{L}_d \) measures the degree of flocking among the C-S particles and fluid.

4.2. **Temporal-variation of \( \mathcal{L} \).** We present a time-decay estimate of the Lyapunov functional introduced in the previous subsection. First, recall our main system:
\[ \partial_t f + \xi \cdot \nabla_x f + \nabla_x (F_u(f) + F_df) = 0, \quad (x, \xi) \in \mathbb{T}^3 \times \mathbb{R}^3, \quad t > 0, \]
\[ \partial_t \rho + \text{div}_x (\rho u) = 0, \]
\[ \partial_t (\rho u) + \text{div}_x (\rho u \otimes u) + \nabla_x p - \mu \Delta_x u - (\mu + \lambda) \nabla_x \text{div} u = - \int_{\mathbb{R}^3} (u(x,t) - \xi) f d\xi. \] (52)

In the sequel, we present the temporal decay estimates of subfunctionals introduced in the previous subsection in a series of lemmas.

**Lemma 4.1.** Let \( [f, \rho, u] \) be any smooth solution to (52). Then, we have
\[ \frac{d}{dt} \left( \int_{\mathbb{T}^3 \times \mathbb{R}^3} \frac{|\xi - \xi_c|^2}{2} f d\xi dx \right) + \psi_m \int_{\mathbb{T}^3 \times \mathbb{R}^3} \frac{|\xi - \xi_c|^2}{2} f d\xi dx \]
\[ \leq \int_{\mathbb{T}^3 \times \mathbb{R}^3} (u - \xi) \cdot (\xi - \xi_c) f d\xi dx. \] (53)

**Proof.** By direct calculation, we have
\[ \frac{d}{dt} \left( \int_{\mathbb{T}^3 \times \mathbb{R}^3} \frac{|\xi - \xi_c|^2}{2} f d\xi dx \right) \]
\[ = - \xi_c(t) \cdot \int_{\mathbb{T}^3 \times \mathbb{R}^3} (\xi - \xi_c(t)) f d\xi dx + \frac{1}{2} \int_{\mathbb{T}^3 \times \mathbb{R}^3} |\xi - \xi_c(t)|^2 \partial_t f d\xi dx \]
\[ := I_{41}^1(t) + I_{41}^2(t). \] (54)

• (Estimate of \( I_{41}^1 \)): It follows from the definition of \( \xi_c \) that we have
\[ I_{41}^1 = 0. \] (55)
(Estimate of $I^{2}_{41}$): We use (52) and the integration by parts to obtain
\[
I^{2}_{41}(t) = \frac{1}{2} \int_{T^3 \times \mathbb{R}^3} |\xi - \xi_c(t)|^2 \partial_t f dx d\xi
\]
\[
= \int_{T^3 \times \mathbb{R}^3} (\xi - \xi_c) \cdot (F_a(f)f + F_d f) dx d\xi
\]
\[
= \int_{\mathbb{T}^3 \times \mathbb{R}^3} \psi(x,y)(\xi - \xi_c) \cdot (\xi - \xi_c) f(x,\xi) f(x,\xi_c) d\xi d\xi dy dx
\]
\[
+ \int_{T^3 \times \mathbb{R}^3} (\xi - \xi_c) \cdot (u - \xi) f dx d\xi.
\]
The first term in the R.H.S. of the relation can be treated using the skew-symmetry of $\xi - \xi_*$ in the transformation $\xi \leftrightarrow \xi_*$ as follows.
\[
\int_{\mathbb{T}^6 \times \mathbb{R}^6} \psi(x,y)\xi \cdot (\xi_* - \xi_*) f(y,\xi_*) f(x,\xi) d\xi_* d\xi dy dx
\]
\[
- \xi_c \cdot \left[ \int_{\mathbb{T}^6 \times \mathbb{R}^6} \psi(x,y)(\xi - \xi_c) f(y,\xi_*) f(x,\xi) d\xi_* d\xi dy dx \right]
\]
\[
= -\frac{1}{2} \int_{\mathbb{T}^6 \times \mathbb{R}^6} \psi(x,y)\xi - \xi_*|^2 f(x,\xi) f(y,\xi_*) d\xi_* d\xi dy dx
\]
\[
\leq -\frac{\psi_m}{2} \int_{\mathbb{T}^6 \times \mathbb{R}^6} |\xi - \xi_*|^2 f(x,\xi) f(y,\xi_*) d\xi_* d\xi dy dx
\]
\[
= -\frac{\psi_m}{2} \left[ \int_{\mathbb{T}^6 \times \mathbb{R}^6} |\xi - \xi_c(t)|^2 + |\xi_* - \xi_c(t)|^2 \right] f(x,\xi) f(y,\xi_*) d\xi_* d\xi dy dx
\]
\[
+ \int_{\mathbb{T}^6 \times \mathbb{R}^6} (\xi - \xi_c(t)) \cdot (\xi_* - \xi_c(t)) f(x,\xi) f(y,\xi_*) d\xi_* d\xi dy dx
\]
\[
= -2\psi_m L_p(t).
\]
Thus, we have
\[
I^{2}_{41}(t) \leq -2\psi_m L_p(t) + \int_{T^3 \times \mathbb{R}^3} (\xi - \xi_c) \cdot (u - \xi) f dx d\xi. \tag{56}
\]
In (54), we combine (55) and (56) to obtain the desired result. \hfill \square

In the sequel, we use the simple notation $(\rho u)_c := \int_{T^3} \rho u dx$.

**Lemma 4.2.** Let $[f, \rho, u]$ be any smooth solution to (52). Then, we have
\[
(i) \quad \frac{d}{dt} \left( \int_{T^3} \frac{\rho}{2} |u - \xi_c|^2 dx + \frac{1}{\gamma - 1} \int_{T^3} \rho dx \right)
\]
\[
= -\mu \int_{T^3} |\nabla u|^2 dx - (\lambda + \mu) \int_{T^3} |\nabla \cdot u|^2 dx
\]
\[
- \int_{T^3 \times \mathbb{R}^3} (u - \xi_c) \cdot (u - \xi) f d\xi dx - \dot{\xi}_c \cdot ((\rho u)_c - \rho_c \dot{\xi}_c). \tag{57}
\]
By Lemma 3.1 and direct estimates, we have

\[ \int_{T_T^3} \rho |u - u_c|^2 dx + \frac{1}{\gamma - 1} \int_{T_T^3} p dx \]

\[ = -\mu \int_{T_T^3} |\nabla u|^2 dx - (\lambda + \mu) \int_{T_T^3} |\nabla \cdot u|^2 dx \]

\[ - \int_{T_T^3 \times \mathbb{R}^3} (u - u_c) \cdot (u - \xi) f d\xi dx - \dot{u}_c \cdot ((\rho u)_c - \rho_c u_c). \]

**Proof.** By direct calculation, we have

\[ \frac{d}{dt} \int_{T_T^3} \rho \frac{1}{2} |u - \xi|^2 dx \]

\[ = \frac{1}{2} \int_{T_T^3} \rho u |u - \xi|^2 dx + \int_{T_T^3} \rho (u - \xi) \cdot (u_t - \dot{\xi}) dx \]

\[ = \frac{1}{2} \int_{T_T^3} \rho u |u - \xi|^2 dx + \int_{T_T^3} \rho (u - \xi) \cdot u_t dx - \dot{\xi} \cdot ((\rho u)_c - \rho_c \xi_c) \]

\[ := T_{42}^1 + T_{42}^2 - \dot{\xi} \cdot ((\rho u)_c - \rho_c \xi_c). \]

• (Estimate of $T_{42}^1$): We use (52) to obtain

\[ T_{42}^1 = -\frac{1}{2} \int_{T_T^3} \nabla \cdot ((\rho u)_c) |u - \xi|^2 dx = \int_{T_T^3} (u - \xi) \cdot (\rho u) \cdot \nabla u dx. \]

• (Estimate of $T_{42}^2$): We split $T_{42}^2$ into two parts:

\[ T_{42}^2 = \int_{T_T^3} \rho u \cdot u_t dx - \xi_c \cdot \int_{T_T^3} \rho u_t dx := T_{42}^{21} + T_{42}^{22}. \]

It should be noted that it follows from (1) and (1) that

\[ \rho u_t = -\rho u \cdot \nabla u - \nabla p + \mu \Delta u + (\mu + \lambda) \nabla (\nabla \cdot u) - \int_{\mathbb{R}^3} (u - \xi) f d\xi. \]

• (Estimate of $T_{42}^{21}$): It follows from (1) that we have

\[ T_{42}^{21} = -\int_{T_T^3} u \cdot \rho u \cdot \nabla u dx - \int_{T_T^3} u \cdot \nabla p dx + (\lambda + \mu) \int_{T_T^3} u \cdot \Delta u dx \]

\[ + \int_{T_T^3} u \cdot \nabla u dx - \int_{T_T^3 \times \mathbb{R}^3} u \cdot (u - \xi) f d\xi dx \]

\[ := \sum_{i=1}^5 T_{42}^{21k}. \]

By Lemma 3.1 and direct estimates, we have

\[ T_{42}^{12} = -\frac{1}{\gamma - 1} \frac{d}{dt} \int_{T_T^3} p dx, \]

\[ T_{42}^{13} = -\mu \int_{T_T^3} |\nabla u|^2 dx, \]

\[ T_{42}^{14} = -(\lambda + \mu) \int_{T_T^3} |\text{div} u|^2 dx. \]

• (Estimate of $T_{42}^{22}$): We use the system (62) to find

\[ T_{42}^{22} = \xi_c \cdot \left( \int_{T_T^3} \rho u \cdot \nabla u dx + \int_{T_T^3} \nabla p dx - \mu \int_{T_T^3} \Delta u dx \right. \]

\[ \left. - (\lambda + \mu) \int_{T_T^3} \nabla (\text{div} u + \int_{T_T^3 \times \mathbb{R}^3} (u - \xi) f d\xi dx \right) \]

\[ = \xi_c \cdot \int_{T_T^3} \rho u \cdot \nabla u dx + \xi_c \cdot \int_{T_T^3 \times \mathbb{R}^3} (u - \xi) f d\xi dx. \]
Lemma 4.3. The following estimates hold.

(i) \(-u'_c \cdot ((\rho u)_c - \rho_c u_c) = \frac{1}{2 \rho_c} \frac{d}{dt} |(\rho u)_c - \rho_c u_c|^2 + \frac{\xi'_e}{\rho_c} \cdot ((\rho u)_c - \rho_c \xi_c) - \xi'_c \cdot (u_e - \xi_c)\).

(ii) \(-\xi'_c \cdot ((\rho u)_c - \rho_c \xi_c) = \frac{1}{2(1 + \rho_c)} \frac{d}{dt} |(\rho u)_c - \rho_c \xi_c|^2\).

Proof. We first note that Proposition 1 (ii), (12) and the elementary addition and subtraction technique imply

\((\rho u)'_c = -\xi'_e, \quad \rho_c u_c = \rho_c \xi_c - \rho_c (\xi_c - u_c)\).

The above relations yield

\[
\frac{d}{dt}|(\rho u)_c - \rho_c u_c|^2 = -\xi'_e \cdot ((\rho u)_c - \rho_c \xi_e) - \xi'_c \cdot (\xi_c - \xi_e) - \xi'_c \cdot (u_e - \xi_c) - \xi'_c \cdot ((\rho u)_c - \rho_c u_c).
\]

We now divide the equation (66) by \(2\rho_c\) to obtain

\[
\frac{1}{2 \rho_c} \frac{d}{dt} |(\rho u)_c - \rho_c u_c|^2 = -\frac{\xi'_e}{\rho_c} \cdot ((\rho u)_c - \rho_c \xi_c) - \xi'_c \cdot (\xi_c - u_c) - \xi'_c \cdot ((\rho u)_c - \rho_c u_c).
\]

Similarly, we obtain

\[
\frac{d}{dt}|(\rho u)_c - \rho_c \xi_c|^2 = 2((\rho u)_c - \rho_c \xi_c) \cdot ((\rho u)'_c - \rho_c \xi'_c) = 2((\rho u)_c - \rho_c \xi_c) \cdot (-\xi'_c - \rho_c \xi'_e) = -2 \frac{1}{1 + \rho_c} \xi'_c \cdot ((\rho u)_c - \rho_c \xi_c).
\]

Again, we have

\[
\frac{1}{2(1 + \rho_c)} \frac{d}{dt} |(\rho u)_c - \rho_c \xi_c|^2 = -\xi'_c \cdot ((\rho u)_c - \rho_c \xi_c).
\]
Remark 3. Note that (i) and (ii) of Lemma 4.3 and the relation

\[
\frac{d}{dt}(|\rho_c \xi_c - \rho_c u_c|^2) = \frac{d}{dt}((\rho u)_c - \rho_c \xi_c|^2) + \frac{d}{dt}((\rho u)_c - \rho_c u_c|^2) + 2 \frac{d}{dt}[(\rho_c \xi_c - (\rho u)_c) \cdot ((\rho u)_c - \rho_c u_c)].
\]

imply

\[
- \xi'_c \cdot ((\rho u)_c - \rho_c \xi_c) - \rho_c u'_c \cdot ((\rho u)_c - \rho_c u_c) = \frac{1}{2(1 + \rho_c)} \frac{d}{dt}((\rho u)_c - \rho_c \xi_c|^2 + \frac{1}{2} \frac{d}{dt}|(\rho u)_c - \rho_c u_c|^2
\]

\[
+ \xi'_c \cdot ((\rho u)_c - \rho_c \xi_c) - \rho_c \xi'_c \cdot (u_c - \xi_c)
\]

\[
= \left( \frac{1}{2} - \frac{1}{2(1 + \rho_c)} \right) \frac{d}{dt}|(\rho u)_c - \rho_c u_c|^2 + \frac{\rho_c^2}{2(1 + \rho_c)} \frac{d}{dt} |\xi_c - u_c|^2 \right) \right.
\]

\[
- \frac{1}{1 + \rho_c} \frac{d}{dt} \left[ (\rho_c \xi_c - (\rho u)_c) \cdot ((\rho u)_c - \rho_c u_c) \right]
\]

\[
+ \xi'_c \cdot ((\rho u)_c - \rho_c \xi_c) - \rho_c \xi'_c \cdot (u_c - \xi_c),
\]

We next study the energy estimate for the system (52) by considering the linear combination

\[
(53) + (57) + \rho_c \times (58)
\]

of the differential inequalities (53) - (58):

\[
\frac{d}{dt} \left( \frac{1}{2} \int_{\mathbb{T}^3 \times \mathbb{R}^3} |\xi - \xi_c|^2 f d\xi dx + \frac{1}{2} \int_{\mathbb{T}^3} \rho|u - \xi_c|^2 dx \right)
\]

\[
+ \frac{\rho_c}{2} \int_{\mathbb{T}^3} \rho|u - u_c|^2 dx + \frac{(1 + \rho_c)}{\gamma - 1} \int_{\mathbb{T}^3} \rho|u|^2 dx
\]

\[
\leq -(1 + \rho_c) \mu \int_{\mathbb{T}^3} |\nabla u|^2 dx - (1 + \rho_c)(\lambda + \mu) \int_{\mathbb{T}^3} |\nabla \cdot u|^2 dx - \int_{\mathbb{T}^3 \times \mathbb{R}^3} |u - \xi_l|^2 f d\xi dx
\]

\[
- \psi_m \int_{\mathbb{T}^3 \times \mathbb{R}^3} |\xi - \xi_c|^2 f d\xi dx - \rho_c \int_{\mathbb{T}^3 \times \mathbb{R}^3} (u - u_c) \cdot (u - \xi) f d\xi dx
\]

\[
- \xi'_c \cdot ((\rho u)_c - \rho_c \xi_c) - \rho_c u'_c \cdot ((\rho u)_c - \rho_c u_c),
\]

We now use the relation

\[
\frac{1}{2} \int_{\mathbb{T}^3} \rho|u - \xi_c|^2 dx = \frac{1}{2} \int_{\mathbb{T}^3} \rho|u - u_c|^2 dx + \frac{\rho_c}{2} |u_c - \xi_c|^2 + \int_{\mathbb{T}^3} \rho(u - u_c) \cdot (u_c - \xi_c) dx
\]

and combine (67) and (68) to obtain the following dissipation estimate.

\[
\frac{dE}{dt} + D \leq 0.
\]

where, the energy functional \( E \) and its dissipation rate functional \( D \) are given as

\[
E := \frac{1}{2} \int_{\mathbb{T}^3 \times \mathbb{R}^3} |\xi - \xi_c|^2 f d\xi dx + \frac{1}{2} (1 + \rho_c) \int_{\mathbb{T}^3} \rho|u - u_c|^2 dx + \frac{(1 + \rho_c)}{\gamma - 1} \int_{\mathbb{T}^3} \rho p dx
\]
Then, the relations between the norms of $L$ functional

Although we can show that the energy functional $E$

Lemma 4.4. [4, 5, 16, 17] There exists a linear operator $B : \left\{ f \in H^2(T^3) \right\} \to [H^3(T^3)]^3$ and a generic constant $C$ independent of $f$ such that

1. $B[f]$ is solution to the problem (70) and linear operator satisfying

$$\|B[f]\|_{H_0^1(T^3)} \leq C\|f\|_{L^2(T^3)}.$$ 

2. If a function $f \in H^1(T^3)$ can be written in the form $f = \nabla \cdot g$ with $g \in (H^2(T^3))^3$, then

$$\|B[f]\|_{L^2(T^3)} \leq C\|g\|_{L^2(T^3)}.$$ 

We next recall the following lemma.
Lemma 4.5. Let $r_0, \bar{r} > 0$ and $\gamma > 1$ be given constants, and set

$$f(r) = r \int_{r_0}^{r} \frac{h^\gamma - r_0^\gamma}{h^2} dh,$$

for $r \in [0, \bar{r}]$. Then, there exist positive constants $C_1$ and $C_2$ such that

$$C_1 (r_0, \bar{r}) (r - r_0)^2 \leq f(r) \leq C_2 (r_0, \bar{r}) (r - r_0)^2 \quad \text{for all } r \in [0, \bar{r}].$$

Lemma 4.6. The following estimates hold:

(i) $\frac{d}{dt} \int_{T^3} \frac{p(\rho)}{\gamma - 1} dx = \frac{d}{dt} \int_{T^3} \rho \int_{p_c}^{\rho} \frac{h^\gamma - \rho_c^\gamma}{h^2} dh dx.$

(ii) $\frac{d}{dt} \left[ \sigma \int_{T^3} \rho (u - \xi_c) \mathcal{B}[\rho - \rho_c] dx \right]$

\begin{align*}
= & \sigma \int_{T^3} \partial_t (\rho \xi_c) \mathcal{B}[\rho - \rho_c] dx - \sigma \int_{T^3} \rho (u - \xi_c) \mathcal{B}[\nabla \cdot (\rho u)] dx \\
& + \sigma \int_{T^3} (\rho u \otimes u) : \nabla \mathcal{B}[\rho - \rho_c] dx + \sigma \int_{T^3} (\rho \gamma - \rho_c^\gamma) (\rho - \rho_c) dx \\
& - \mu \sigma \int_{T^3} \nabla u : \nabla \mathcal{B}[\rho - \rho_c] dx - \sigma (\mu + \lambda) \int_{T^3} \nabla \cdot (u (\rho - \rho_c)) dx \\
& - \sigma \int_{\mathbb{R}^3} (u - \xi) f \cdot \mathcal{B}[\rho - \rho_c] d\xi dx \\
:= & \sigma \mathcal{D}_1.
\end{align*}

Proof. (i) Recall that $p(\rho) = \rho^\gamma$ and $\rho_c(t) = \rho_c(0)$, $t > 0$. Then, by direct calculation, we have

$$\int_{p_c}^{\rho} \frac{h^\gamma - \rho_c^\gamma}{h^2} dh = \frac{\rho^{\gamma - 1}}{\gamma - 1} + \frac{\rho_c^{\gamma - 1}}{\gamma - 1} - \frac{\rho_c^{\gamma - 1}}{\gamma - 1}.$$

Then, we multiply $\rho$ by the above relation and integrate the resulting relation over $T^3$ to get

$$\int_{T^3} \rho \left[ \int_{p_c}^{\rho} \frac{h^\gamma - \rho_c^\gamma}{h^2} dh \right] dx = \int_{T^3} \frac{\rho^{\gamma - 1}}{\gamma - 1} dx - \frac{\rho_c^{\gamma - 1}}{\gamma - 1}.$$

Since $\rho_c$ is constant, the above relation implies

$$\frac{d}{dt} \int_{T^3} \frac{\rho^{\gamma - 1}}{\gamma - 1} dx = \frac{d}{dt} \int_{T^3} \rho \left[ \int_{p_c}^{\rho} \frac{h^\gamma - \rho_c^\gamma}{h^2} dh \right] dx.$$

(ii) We use the momentum equations (52) to obtain

$$\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p - \mu \Delta u - (\mu + \lambda) \nabla \text{div} u = \int_{\mathbb{R}^3} (\xi - u) f d\xi,$$

$$\partial_t \mathcal{B}[\rho - \rho_c] = \mathcal{B}[\rho_c],$$

to obtain

$$\frac{d}{dt} \left[ \sigma \int_{T^3} \rho (u - \xi_c) \mathcal{B}[\rho - \rho_c] dx \right]$$

\begin{align*}
= & \sigma \int_{T^3} (\rho u - \xi_c) \mathcal{B}[\rho - \rho_c] dx + \sigma \int_{T^3} \rho (u - \xi_c) \mathcal{B}[\rho_c] dx + \sigma \int_{T^3} (\rho \xi_c) \mathcal{B}[\rho - \rho_c] dx \\
:= & \mathcal{D}_1 + \mathcal{D}_2 + \sigma \int_{T^3} (\rho \xi_c) \mathcal{B}[\rho - \rho_c] dx.
\end{align*}

- (Estimate of $\mathcal{D}_1$): We use the momentum equations

$$\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p - \mu \Delta u - (\mu + \lambda) \nabla \text{div} u = \int_{\mathbb{R}^3} (\xi - u) f d\xi$$
to find

\[
I_{43}^1 = -\sigma \int_{T^3} \nabla \cdot (\rho u \otimes u) B[\rho - \rho_c] dx - \sigma \int_{T^3} \nabla p B[\rho - \rho_c] dx \\
+ \mu \sigma \int_{T^3} \Delta u B[\rho - \rho_c] dx + (\mu + \lambda) \sigma \int_{T^3} \nabla (\nabla \cdot u) B[\rho - \rho_c] dx \\
- \sigma \int_{\mathbb{R}^3 \times T^3} (u - \xi) f B[\rho - \rho_c] d\xi dx
\]

\[
=: I_{43}^{11} + I_{43}^{12} + I_{43}^{13} + I_{43}^{14} + I_{43}^{15}.
\]

Then, the terms \( I_{43}^k \) can be estimated as follows.

\( \diamond \) (Estimate of \( I_{43}^{11} \)): We use integration by parts to find

\[
I_{43}^{11} = \sigma \int_{T^3} \rho u \otimes u : \nabla B[\rho - \rho_c] dx.
\]

\( \diamond \) (Estimate of \( I_{43}^{12} \)): We use \( p(\rho) = \rho^\gamma \) and integration by parts to obtain

\[
I_{43}^{12} = \sigma \int_{T^3} \rho^\gamma \nabla \cdot B(\rho - \rho_c) dx = \sigma \int_{T^3} \rho^\gamma (\rho - \rho_c) dx = \sigma \int_{T^3} (\rho^\gamma - \rho_c^\gamma)(\rho - \rho_c) dx.
\]

\( \diamond \) (Estimate of \( I_{43}^{13} \)): In this case, we have

\[
I_{43}^{13} = -\mu \sigma \int_{T^3} \nabla u : \nabla B(\rho - \rho_c) dx.
\]

\( \diamond \) (Estimate of \( I_{43}^{14} \)): We use the relation \( \nabla \cdot B(f) = f \) to obtain

\[
I_{43}^{14} = -(\mu + \lambda) \sigma \int_{T^3} \nabla \cdot u \nabla B(\rho - \rho_c) dx = -(\mu + \lambda) \sigma \int_{T^3} \nabla \cdot u (\rho - \rho_c) dx.
\]

\( \bullet \) (Estimate of \( I_{43}^2 \)): We use the continuity equation \( \rho_t = -\nabla \cdot (\rho u) \) to get

\[
I_{43}^2 = -\sigma \int_{T^3} \rho (u - \xi_c) B(\nabla \cdot (\rho u)) dx.
\]

We next introduce perturbed functionals \( E^\sigma \) and dissipation rate \( D^\sigma \) as follows:

\[
E^\sigma := E - \sigma \int_{T^3} \rho (u - \xi_c) B(\rho - \rho_c) dx, \quad D^\sigma := D + \sigma D_1.
\]  \tag{72}

Here, the perturbation \( D_1 \) appears in Lemma 4.6.

We next show the equivalence between the energy functional \( E^\sigma \) and \( L \) for sufficiently small \( \sigma \) and initial data. Then, it follows from (69), Lemma 4.7 (ii) and (72) that

\[
\frac{dE^\sigma}{dt} + D^\sigma \leq 0.
\]  \tag{73}

**Lemma 4.7.** Under the assumption of Theorem 2.4, there exist positive constants \( \bar{C}_0 \) and \( \bar{C}_1 \) independent of \( t \) such that

\[
\bar{C}_0 L(t) \leq E^\sigma(t) \leq \bar{C}_1 L(t), \quad 0 \leq t \leq T.
\]
Proof. We first notice that
\[
\mathcal{E}^\sigma = \frac{1}{2} \int_{T^3 \times \mathbb{R}^3} |\xi - \xi_c|^2 f d\xi d\xi + \frac{1}{2}(1 + \rho_c) \int_{T^3} \rho |u - u_c|^2 dx + \frac{1 + \rho_c}{\gamma - 1} \int_{T^3} \rho^\gamma dx \\
+ \frac{\rho_c}{2(1 + \rho_c)} |u_c - \xi_c|^2 - \frac{2 + \rho_c}{2(1 + \rho_c)} (\rho u)_c - \rho_{c} u_c|^2 \\
+ \frac{1}{1 + \rho_c} ((\rho u)_c - \rho_c u_c) \cdot (u_c - \xi_c) - \sigma \int_{T^3} \rho (u - \xi_c) B[\rho - \rho_c] d\xi,
\]
due to
\[
\frac{1}{1 + \rho_c} (\rho \xi_c - (\rho u)_c) \cdot ((\rho u)_c - \rho_c u_c) + ((\rho u)_c - \rho_c u_c) \cdot (u_c - \xi_c) \\
= \frac{1}{1 + \rho_c} \left( \left( (\rho u)_c - \rho_c u_c \right) \cdot (u_c - \xi_c) - \left( (\rho u)_c - \rho_c u_c \right)^2 \right),
\]
To show the equivalence of \( L \) and \( \mathcal{E}^\sigma \), we estimate as follows.

- **Case A** (potential energy term \( \int_{T^3} p d\xi \)): We use Lemma 4.6 and Lemma 4.7(i) to conclude
  \[
  C_* \int_{T^3} (\rho - \rho_c)^2 dx \leq \int_{T^3} \rho \left( \int_{\rho_c}^{\rho} \frac{\hbar^2 - \rho^2}{h^2} d\rho \right) dx \\
  \leq C^* \int_{T^3} (\rho - \rho_c)^2 dx,
  \]
  where \( C_*, C^* \) are positive constants depending on \( \rho_c, \| \rho \|_{L^\infty} \) and \( \gamma \). In particular, \( C_* \) satisfies
  \[
  C_* < \frac{\gamma}{2} \rho_c^{-2},
  \]
  by Lemma 4.5 from [15].

- **Case B** (Perturbation term): We use Lemma 4.5 to obtain
  \[
  \left| \int_{T^3} -\sigma \rho (u - \xi_c) B[\rho - \rho_c] d\xi \right| \\
  \leq \sigma \| \rho \|_{L^\infty} \int_{T^3} \rho |u - \xi_c|^2 dx + \frac{C\sigma}{2} \int_{T^3} (\rho - \rho_c)^2 dx \\
  \leq \sigma \| \rho \|_{L^\infty} \left( \int_{T^3} \rho |u - u_c|^2 dx + \rho_c |u_c - \xi_c|^2 \right) + \frac{C\sigma}{2} \int_{T^3} (\rho - \rho_c)^2 dx.
  \]

- **Case C** (Terms involving local averages): Straightforward computations yield that
  \[
  |(\rho u)_c - \rho_c u_c|^2 = \left( \int_{T^3} \rho (u - u_c) dx \right)^2 \leq \rho_c \int_{T^3} \rho |u - u_c|^2 dx,
  \]
  and
  \[
  \frac{1}{1 + \rho_c} |(\rho u)_c - \rho_c u_c) \cdot (u_c - \xi_c)| \\
  \leq \frac{\delta_1}{2(1 + \rho_c)} |u_c - \xi_c|^2 + \frac{1}{2\delta_1 (1 + \rho_c)} |(\rho u)_c - \rho_c u_c|^2 \\
  \leq \frac{\delta_1}{2(1 + \rho_c)} |u_c - \xi_c|^2 + \rho_c \left( \frac{1}{2\delta_1 (1 + \rho_c)} - \delta_2 \right) \int_{T^3} \rho |u - u_c|^2 dx \\
  + \delta_2 \| u \|_{T^3 \times (0, \infty)}^2 \int_{T^3} (\rho - \rho_c)^2 dx,
  \]
where \( \delta_1, \delta_2 > 0 \) satisfy \( 1 > 2\delta_1\delta_2(1 + \rho_c) \) and we used

\[
| (\rho u)_c - \rho_c u_c |^2 = \left| \int_{T^3} (\rho - \rho_c) u dx \right|^2 \leq \| u \|_{L^\infty(0, \infty; L^2)}^2 \int_{T^3} (\rho - \rho_c)^2 dx.
\]

Thus, we obtain

\[
\mathcal{E}^\sigma(t) \geq \frac{1}{2} \int_{T^3 \times \mathbb{R}^3} |\xi - \xi_c|^2 f d\xi dx + c_1(\rho, \delta_1, \delta_2, \sigma) \int_{T^3} \rho |u - u_c|^2 dx
\]

\[
+ c_2(\rho, u, \delta_2, \sigma) \int_{T^3} (\rho - \rho_c)^2 dx + c_3(\rho, \delta_1, \sigma) |u_c - \xi_c|^2,
\]

where

\[
c_1(\rho, \delta_1, \delta_2, \sigma) := \frac{1 + \rho_c}{2} - \frac{\rho_c(\rho_c + 2)}{2(1 + \rho_c)} - \rho_c \left( \frac{1}{2\delta_1(1 + \rho_c)} - \delta_2 \right) - \sigma \| \rho \|_{L^\infty},
\]

\[
= \frac{\delta_1 - \rho_c}{2\delta_1(1 + \rho_c)} + \rho_c \delta_2 - \sigma \| \rho \|_{L^\infty},
\]

\[
c_2(\rho, u, \delta_2, \sigma) := C_\sigma - \delta_2 \| u \|_{L^\infty(0, \infty; L^2)} - C \sigma,
\]

\[
c_3(\rho, \delta_1, \sigma) := \frac{\rho_c - \delta_1}{2(1 + \rho_c)} - \sigma \| \rho \|_{L^\infty} \rho_c,
\]

where \( \delta_1, \delta_2, \sigma > 0 \) are determined below. We begin by choosing \( \delta_1 = A\rho_c \) where \( A \in (0, 1) \) satisfies

\[
\frac{1}{\rho_c(1 + \rho_c)} \left( \frac{1}{A} - 1 \right) < \min \left\{ \frac{C_\sigma}{2\| u \|_{L^\infty(0, \infty; L^2)}}, \frac{1}{2A\rho_c(1 + \rho_c)} \right\}. \tag{76}
\]

This is possible since the left hand side of (76) goes to 0 as \( A \to 1^- \). This deduces that (76) holds for the constant \( A \) sufficiently close to 1. This enables us to select the constant \( \delta_2 \) such that

\[
\frac{1}{\rho_c(1 + \rho_c)} \left( \frac{1}{A} - 1 \right) < \delta_2 < \min \left\{ \frac{C_\sigma}{2\| u \|_{L^\infty(0, \infty; L^2)}}, \frac{1}{2A\rho_c(1 + \rho_c)} \right\}.
\]

Then we find that \( \delta_2 \) satisfies

\[
\frac{\rho_c \delta_2}{2} > \frac{\rho_c - \delta_1}{2\delta_1(1 + \rho_c)}, \quad \frac{C_\sigma}{2} > \delta_2 \| u \|_{L^\infty(0, \infty; L^2)}^2, \quad \text{and} \quad 1 > 2\delta_1(1 + \rho_c)\delta_2. \tag{77}
\]

We finally choose \( \sigma > 0 \) such that

\[
0 < \sigma < \min \left\{ \frac{\rho_c - \delta_1}{2\rho_c \| \rho \|_{L^\infty(1 + \rho_c)}}, \frac{C_\sigma}{2C'}, \frac{\rho_c \delta_2}{2\| \rho \|_{L^\infty}} \right\}. \tag{78}
\]

Then we use the relations for \( \delta_1, \delta_2, \sigma \) in (76), (77), and (78) to conclude that \( c_1, c_2, c_3 > 0 \) and \( \mathcal{E}^\sigma(t) > L(t) \) for all \( t \geq 0 \). For the upper bound of \( \mathcal{E}^\sigma(t) \), we just choose the maximum value of each coefficient of \( \mathcal{E}^\sigma(t) \) using the estimates (74). This completes the proof.

We are now ready to provide the proof of Theorem 2.4.
4.4. **Proof of Theorem 2.4.** First, we need to bind $\mathcal{E}^\sigma$ by $\mathcal{D}^\sigma$ in order to obtain the exponential decay of $\mathcal{E}^\sigma$ from (73). We can estimate

\[
\mathcal{D}^\sigma \geq C_1(\mu, \sigma, \rho, u, f) \int_{T^3} |\nabla u|^2 dx + C_2(\mu, \sigma, \rho, u, f) \int_{T^3} (\rho - \rho_c)^2 dx \\
+ C_3(\mu, \sigma, \rho, u, f) \int_{T^3 \times \mathbb{R}^3} |u - \xi|^2 f d\xi dx + \psi_m \int_{T^3 \times \mathbb{R}^3} |\xi - \xi_c|^2 f d\xi dx
\]

(79)

where $C_1, C_2, C_3$ are positive constants for sufficiently small $\sigma, \|\rho\|_{L^\infty}$ and $\|u\|_{L^\infty}$:

\[
C_1(\mu, \sigma, \rho, u, f) := \frac{3}{4} \mu + \rho_c - C\rho_c\|\rho_p\|_{L^\infty} - C\sigma\|\rho\|_{L^\infty} \\
- C\|\rho\|_{L^\infty} - C\|\rho\|_{L^\infty}^2 - C\|\rho\|_{L^\infty}\|u\|_{L^\infty},
\]

\[
C_2(\mu, \sigma, \rho, u, f) := C\sigma \left[1 - C\|\rho\|_{L^\infty} - C\|u\|_{L^\infty} - C\sigma\|\rho\|_{L^\infty} - C\|u\|_{L^\infty}^2 - C\sigma\|f\|_{L^\infty} - C\mu\sigma \\
- C\sigma(\lambda + \mu) - C\sigma^{-1}(\|u\|_{L^\infty}^2 + \|\rho\|_{L^\infty}\|u\|_{L^\infty} + \|\rho\|_{L^\infty}) \right],
\]

\[
C_3(\mu, \sigma, \rho, u, f) := \frac{1}{2} - C\rho_c - C\|\rho\|_{L^\infty}.
\]

Here, $\rho_p$ is the local mass density of C-S particles, i.e., $\rho_p := \int_{\mathbb{R}^3} f d\xi$. We postpone the proof of the estimate (81) to Appendix A, since it is rather lengthy and technical. We next calculate the dissipation term of $|u_c - \xi_c|^2$. For this purpose, it should be noted that

\[
\int_{T^3 \times \mathbb{R}^3} |u - \xi|^2 f d\xi dx
\]

\[
= \int_{T^3 \times \mathbb{R}^3} |u - u_c|^2 f d\xi dx + \int_{T^3 \times \mathbb{R}^3} |u_c - \xi_c|^2 f d\xi dx + \int_{T^3 \times \mathbb{R}^3} |\xi_c - \xi|^2 f d\xi dx \\
+ 2 \int_{T^3 \times \mathbb{R}^3} (u - u_c) \cdot (u_c - \xi_c) f d\xi dx + 2 \int_{T^3 \times \mathbb{R}^3} (u - u_c) \cdot (\xi_c - \xi) f d\xi dx.
\]

Then, the last two terms in the R.H.S. of the above relation can be estimated as

\[
\left| 2 \int_{T^3 \times \mathbb{R}^3} (u - u_c) \cdot (u_c - \xi_c) f d\xi dx \right|
\]

\[
\leq 2\|\rho_p\|_{L^\infty} \int_{T^3} |u - u_c|^2 dx + \frac{1}{2} |u_c - \xi_c|^2
\]

\[
\leq 2\|\rho_p\|_{L^\infty} \int_{T^3} |\nabla u|^2 dx + \frac{1}{2} |u_c - \xi_c|^2,
\]

\[
\left| 2 \int_{T^3 \times \mathbb{R}^3} (u - u_c) \cdot (\xi_c - \xi) f d\xi dx \right|
\]

\[
\leq 2\|\rho_p\|_{L^\infty} \int_{T^3} |u - u_c|^2 dx + \frac{1}{2} \int_{T^3 \times \mathbb{R}^3} |\xi_c - \xi|^2 f d\xi dx
\]

\[
\leq 2\|\rho_p\|_{L^\infty} \int_{T^3} |\nabla u|^2 dx + \frac{1}{2} \int_{T^3 \times \mathbb{R}^3} |\xi_c - \xi|^2 f d\xi dx.
\]
This yields
\[
\int_{T^3 \times \mathbb{R}^3} |u - \xi|^2 f d\xi dx \\
\geq -4\|\rho_p\|_{L^\infty} \int_{T^3} |\nabla u|^2 dx + \frac{1}{2} |u_c - \xi_c|^2 + \frac{1}{2} \int_{T^3 \times \mathbb{R}^3} |\xi - \xi|^2 f d\xi dx.
\]
Hence, we have
\[
\mathcal{D}^\sigma \geq (C_1(\mu, \sigma, \rho, u, f) - 4C_3(\mu, \sigma, \rho, u, f)\|\rho_p\|_{L^\infty}) \int_{T^3} |\nabla u|^2 dx \\
+ C_2(\mu, \sigma, \rho, u, f) \int_{T^3} (\rho - \rho_c)^2 dx + \left(\frac{C_3(\mu, \sigma, \rho, u, f)}{2} + \psi_m\right) \int_{T^3 \times \mathbb{R}^3} |\xi - \xi|^2 f d\xi dx \\
+ \left(\frac{C_3(\mu, \sigma, \rho, u, f)}{2} - C\rho_c - C\|\rho\|_{L^\infty} - C\|\rho\|_{L^2}^2\right) |u_c - \xi_c|^2,
\]
Since \( \mu \) is sufficiently large with \( \mu > \frac{8}{3}\|\rho_p\|_{L^\infty} \), we have
\[
C_1(\mu, \sigma, \rho, u, f) - 4C_3(\mu, \sigma, \rho, u, f)\|\rho_p\|_{L^\infty} > 0.
\]
For sufficiently small \( \|\rho\|_{L^\infty} \), we have the positive constant
\[
\frac{C_3(\mu, \sigma, \rho, u, f)}{2} - C\rho_c - C\|\rho\|_{L^\infty} - C\|\rho\|_{L^2}^2 > 0.
\]
Hence, we have
\[
\mathcal{E}^\sigma \leq C\mathcal{D}^\sigma.
\]
This and (73) yield
\[
\frac{d}{dt}\mathcal{E}^\sigma + C\mathcal{E}^\sigma \leq 0,
\]
Finally, we use Gronwall’s inequality and Lemma 4.7 to have
\[
\tilde{C}_0\mathcal{L}(t) \leq \mathcal{E}^\sigma(t) \leq \mathcal{E}^\sigma(0)e^{-Ct} \leq \tilde{C}_1\mathcal{L}(0)e^{-Ct}
\]
This completes the proof.

5. Conclusion. We presented a new coupled particle-fluid model for flocking. For the dynamics of the Cucker-Smale particles, we employed the kinetic Cucker-Smale model introduced in [21] and the compressible Navier-Stokes system for fluids. Previously, the authors considered the interaction problem between Cucker-Smale particles and incompressible fluids in the same spirit and studied the global well-posedness of weak and strong solutions to the proposed model and asymptotic flocking estimate in an \textit{a priori} setting. In this paper, we proposed a particle-fluid interaction model constructed by coupling the kinetic Cucker-Smale model and the compressible Navier-Stokes system via the drag forces. We showed that our proposed model has a unique strong solution for smooth initial and small data in suitable Sobolev spaces. We also showed that our model admits a time-asymptotic exponential flocking for smooth solutions using the robust Lyapunov functional approach.
Appendix A. A lower bound estimate of $\mathcal{D}^\sigma$. In this section, we provide a lower bound estimate (81) for the dissipation rate $\mathcal{D}^\sigma$.

First of all, recall that $\mathcal{D}^\sigma$ is given as

$$\mathcal{D}^\sigma(t) := (1 + \rho_c)\mu \int_{T^3} |\nabla u|^2 dx + (1 + \rho_c)(\lambda + \mu) \int_{T^3} |\nabla \cdot u|^2 dx$$

$$+ \int_{T^3 \times \mathbb{R}^3} |u - \xi|^2 f d\xi dx + m_\rho \int_{T^3 \times \mathbb{R}^3} |\xi - \xi_c|^2 f d\xi dx$$

$$+ \rho_c \int_{T^3 \times \mathbb{R}^3} (u - u_c) \cdot (u - \xi) f d\xi dx - \xi' \cdot ((\rho u)_c - \rho_c \xi_c)$$

$$+ \rho_c \xi' \cdot (u_c - \xi_c) + \sigma \int_{T^3} \rho(u - \xi_c) B[\nabla \cdot (\rho u)] dx$$

$$- \sigma \int_{T^3} \partial_t (\rho \xi_c) B[\rho - \rho_c] dx + \sigma \int_{T^3} \rho u \otimes u : \nabla B[\rho - \rho_c] dx$$

$$+ \sigma \int_{T^3} (\rho^\gamma - \rho_c^\gamma)(\rho - \rho_c) dx - \sigma \int_{T^3} \nabla u : \nabla B[\rho - \rho_c] dx$$

$$- \sigma(\lambda + \mu) \int_{T^3} \nabla \cdot u(\rho - \rho_c) dx - \sigma \int_{T^3 \times \mathbb{R}^3} (u - \xi) f B[\rho - \rho_c] d\xi dx.$$

We set

$$\sum_{k=1}^{10} J_k := \rho_c \int_{T^3 \times \mathbb{R}^3} (u - u_c) \cdot (u - \xi) f d\xi dx - \xi' \cdot ((\rho u)_c - \rho_c \xi_c)$$

$$+ \rho_c \xi' \cdot (u_c - \xi_c) + \sigma \int_{T^3} \rho(u - \xi_c) B[\nabla \cdot (\rho u)] dx$$

$$- \sigma \int_{T^3} \partial_t (\rho \xi_c) B[\rho - \rho_c] dx + \sigma \int_{T^3} \rho u \otimes u : \nabla B[\rho - \rho_c] dx$$

$$+ \sigma \int_{T^3} (\rho^\gamma - \rho_c^\gamma)(\rho - \rho_c) dx - \sigma \int_{T^3} \nabla u : \nabla B[\rho - \rho_c] dx$$

$$- \sigma(\lambda + \mu) \int_{T^3} \nabla \cdot u(\rho - \rho_c) dx - \sigma \int_{T^3 \times \mathbb{R}^3} (u - \xi) f B[\rho - \rho_c] d\xi dx.$$

In the sequel, we use the following estimates:

$$|\xi'_c(t)| \leq \left( \int_{T^3 \times \mathbb{R}^3} |u - \xi|^2 f d\xi dx \right)^{\frac{1}{2}}, \quad |\xi_c(t)| \leq (\mathbb{E}(0))^{\frac{1}{2}}. \quad (80)$$

We next estimate for each term $J_k$, $k = 1, \ldots, 10$, separately.

• (Estimate of $J_1$): We use Poincare inequality to obtain

$$J_1 \leq \frac{\rho_c}{2} \int_{T^3 \times \mathbb{R}^3} |u - u_c|^2 f d\xi dx + \frac{\rho_c}{2} \int_{T^3 \times \mathbb{R}^3} |u - \xi|^2 f d\xi dx$$

$$\leq \|\rho_p\|_{L^\infty} \frac{\rho_c}{2} \int_{T^3} |u - u_c|^2 dx + \frac{\rho_c}{2} \int_{T^3 \times \mathbb{R}^3} |u - \xi|^2 f d\xi dx$$

$$\leq \|\rho_p\|_{L^\infty} \frac{\rho_c \pi_3}{2} \int_{T^3} |\nabla u|^2 dx + \frac{\rho_c}{2} \int_{T^3 \times \mathbb{R}^3} |u - \xi|^2 f d\xi dx,$$

where $\rho_p$ and $\pi_3$ are the local mass density of C-S particles and Poincare constant for the domain $\mathbb{T}^3$, respectively.
(Estimate of $J_2$): We use the relation
\[
|\langle \rho \nu \rangle_c - \rho_c \xi_c | = | \int_{T^3} \rho u dx - \int_{T^3} \rho \xi_c(t) dx |
\]
\[
\leq ||\rho||_{L^\infty} \int_{T^3} |u - \xi_c| dx
\]
\[
\leq ||\rho||_{L^\infty} \left( \int_{T^3} |u - \xi_c|^2 dx \right)^{\frac{1}{2}},
\]
to obtain
\[
J_2 \leq ||\rho||_{L^\infty} \left( \frac{|\xi_c|^2}{2} + \frac{1}{2} \int_{T^3} |u - \xi_c|^2 dx \right)
\]
\[
\leq ||\rho||_{L^\infty} \left( \frac{1}{2} \int_{T^3 \times R^3} |u - \xi|^2 f \rho \xi dx + \int_{T^3} |u - u_c|^2 dx + |u_c - \xi_c|^2 \right)
\]
\[
\leq \frac{||\rho||_{L^\infty}}{2} \int_{T^3 \times R^3} |u - \xi|^2 f \rho \xi dx + ||\rho||_{L^\infty} \int_{T^3} |\nabla u|^2 dx + ||\rho||_{L^\infty} |u_c - \xi_c|^2.
\]

(Estimate of $J_3$): We use (80) to obtain
\[
J_3 \leq \frac{\rho_c}{2} \int_{T^3 \times R^3} |u - \xi|^2 f \rho \xi dx + \frac{\rho_c}{2} |u_c - \xi_c|^2.
\]

(Estimate of $J_4$): It follows from Lemma 4.4 that
\[
\int_{T^3} \rho (u - \xi_c) \mathcal{B} \nabla \cdot (\rho \nu) dx
\]
\[
= \int_{T^3} \rho (u - \xi_c) \mathcal{B} \nabla \cdot (\rho (u - \xi_c)) dx + \int_{T^3} \rho (u - \xi_c) \mathcal{B} \nabla \cdot (\rho \xi_c) dx
\]
\[
= \int_{T^3} \rho (u - \xi_c) \mathcal{B} \nabla \cdot (\rho (u - \xi_c)) dx + \int_{T^3} \rho (u - \xi_c) \mathcal{B} \nabla \cdot ((\rho - \rho_c) \xi_c) dx
\]
\[
\leq \int_{T^3} \rho^2 |u - \xi_c|^2 dx + C \int_{T^3} \rho^2 |u - \xi_c|^2 dx + ||\rho||_{L^\infty} \int_{T^3} |u - \xi_c| |\mathcal{B} \nabla \cdot ((\rho - \rho_c) \xi_c)| dx
\]
\[
\leq C ||\rho||_{L^\infty}^2 \left( \int_{T^3} |u - u_c|^2 dx + |u_c - \xi_c|^2 \right) + C ||\rho||_{L^\infty} \left( \int_{T^3} |u - u_c|^2 dx + |u_c - \xi_c|^2 \right)
\]
\[
+ C ||\rho||_{L^\infty} (E(0)) \int_{T^3} (\rho - \rho_c)^2 dx
\]
\[
\leq C \big( ||\rho||_{L^\infty}^2 + ||\rho||_{L^\infty} \big) \left( \int_{T^3} |\nabla u|^2 dx + |u_c - \xi_c|^2 \right) + C ||\rho||_{L^\infty} (E(0)) \int_{T^3} (\rho - \rho_c)^2 dx.
\]

(Estimate of $J_5$): We use the relations
\[
\partial_t (\rho \xi_c) = \xi_c \partial_t \rho + \rho \xi_c' = -\xi_c \nabla_{\rho} \cdot (\rho \nu) + \rho \xi_c',
\]
and
\[
\int_{T^3} \rho_c u_c \xi_c \nabla \mathcal{B} [\rho - \rho_c] dx = 0.
\]
to obtain
\[
\int_{T^3} \partial_t (\rho \xi_c) \mathcal{B} [\rho - \rho_c] dx
\]
\[
= \int_{T^3} \rho (u - u_c) \xi_c \nabla \mathcal{B} [\rho - \rho_c] dx
\]
\[
+ \int_{T^3} (\rho - \rho_c) u_c \xi_c \nabla \mathcal{B} [\rho - \rho_c] dx + \sigma \xi_c' \int_{T^3} \rho \mathcal{B} [\rho - \rho_c] dx
\]
\[
:= J_{51} + J_{52} + J_{53}.
\]
Here, the terms $\mathcal{J}_{6i}$, $i = 1, 2, 3$ can be estimated as

$$
\mathcal{J}_{61} \leq \sigma \langle \mathcal{E}(0) \rangle \frac{1}{2} ||\rho||_{L^\infty} \left( \int_{\Omega^3} |u - u_c|^2 \, dx + \int_{\Omega^3} |
abla B|\rho - \rho_c||^2 \, dx \right)
$$

\[
\leq \sigma \langle \mathcal{E}(0) \rangle \frac{1}{2} ||\rho||_{L^\infty} \left( \int_{\Omega^3} |u - u_c|^2 \, dx + C \int_{\Omega^3} (\rho - \rho_c)^2 \, dx \right)
\]

\[
\leq C \sigma \langle \mathcal{E}(0) \rangle \frac{1}{2} ||\rho||_{L^\infty} \left( \int_{\Omega^3} |\nabla u|^2 \, dx + \int_{\Omega^3} (\rho - \rho_c)^2 \, dx \right),
\]

$$
\mathcal{J}_{62} \leq \sigma \|u\|_{L^\infty} \langle \mathcal{E}(0) \rangle \frac{1}{2} \int_{\Omega^3} (\rho - \rho_c)^2 \, dx,
$$

$$
\mathcal{J}_{63} \leq \sigma ||\rho||_{L^\infty} \int_{\Omega^3} |\xi'| \|B|\rho - \rho_c| \, dx
$$

\[
\leq \frac{\|\rho\|_{L^\infty}}{2} \int_{\Omega^3} |u - \xi|^2 f \, dx + \frac{\|\rho\|_{L^\infty}}{2} \int_{\Omega^3} (\rho - \rho_c)^2 \, dx,
\]

\[
:\text{ (Estimate of } \mathcal{J}_6) \text{: For the related with convection term, we get}
\]

$$
\mathcal{J}_6 = \sigma \int_{\Omega^3} \rho (u - u_c) \otimes u : \nabla B |\rho - \rho_c| \, dx + \sigma \int_{\Omega^3} \rho c \otimes (u - u_c) : \nabla B |\rho - \rho_c| \, dx
$$

\[
+ \sigma \int_{\Omega^3} (\rho - \rho_c) u_c \otimes u_c : \nabla B |\rho - \rho_c| \, dx
\]

\[=: \sigma (\mathcal{J}_{61} + \mathcal{J}_{62} + \mathcal{J}_{63}).\]

Then, the terms $\mathcal{J}_{6i}$, $i = 1, 2, 3$ can be estimated as

$$
\mathcal{J}_{61} \leq \|\rho\|_{L^\infty} \int_{\Omega^3} |u - u_c||u|| \nabla B|\rho - \rho_c|| \, dx
$$

\[
\leq \frac{\|\rho\|_{L^\infty} \|u\|_{L^\infty}}{2} \left( \int_{\Omega^3} |u - u_c|^2 \, dx + \int_{\Omega^3} (\rho - \rho_c)^2 \, dx \right),
\]

$$
\mathcal{J}_{62} \leq \frac{\|\rho\|_{L^\infty} \|u\|_{L^\infty}}{2} \left( \int_{\Omega^3} |u - u_c|^2 \, dx + \int_{\Omega^3} (\rho - \rho_c)^2 \, dx \right),
$$

$$
\mathcal{J}_{63} \leq \|u\|_{L^\infty}^2 \int_{\Omega^3} (\rho - \rho_c)^2 \, dx.
$$

Thus, we have

$$
\sigma \int_{\Omega^3} \rho u \otimes u : \nabla B |\rho - \rho_c| \, dx
$$

\[
\leq \frac{\sigma \|\rho\|_{L^\infty}}{2} \left( \|u\|_{L^\infty} + \|u\|_{L^2(\Omega^3)} \right) \left( \int_{\Omega^3} |u - u_c|^2 \, dx + \int_{\Omega^3} (\rho - \rho_c)^2 \, dx \right)
\]

\[
+ \sigma \|u\|_{L^\infty}^2 \int_{\Omega^3} (\rho - \rho_c)^2 \, dx.
\]
• (Estimates of $J_k$, $k = 7, 8, 9, 10$): By direct estimates, we have

$$J_7 = \sigma \int_{T^3} (\rho^2 - \rho_c^2)(\rho - \rho_c) \, dx \geq \sigma C \int_{T^3} (\rho - \rho_c)^2 \, dx$$

$$J_8 \leq \sigma \mu \int_{T^3} \nabla u : \nabla B(\rho - \rho_c) \, dx \leq \frac{\mu}{4} \int_{T^3} |\nabla u|^2 \, dx + \sigma^2 \mu C \int_{T^3} (\rho - \rho_c)^2 \, dx$$

$$J_9 \leq -\sigma(\lambda + \mu) \int_{T^3} (\nabla \cdot u)(\rho - \rho_c) \, dx$$

$$\leq \frac{\lambda + \mu}{4} \int_{T^3} |\nabla \cdot u|^2 \, dx + \sigma^2(\lambda + \mu) \int_{T^3} (\rho - \rho_c)^2 \, dx$$

$$J_{10} \leq \frac{1}{2} \int_{T^3 \times \mathbb{R}^3} \left| u - \xi \right|^2 f \, dx \, d\xi + \frac{C\sigma^2 ||f||_{L^\infty}}{2} \int_{T^3} (\rho - \rho_c)^2 \, dx.$$ 

Finally, we combine all the estimates to get the desired lower bound estimate of $D^\sigma$:

$$D^\sigma \geq C_1(\mu, \sigma, \rho, u, f) \int_{T^3} |\nabla u|^2 \, dx + C_2(\mu, \sigma, \rho, u, f) \int_{T^3} (\rho - \rho_c)^2 \, dx$$

$$+ C_3(\mu, \sigma, \rho, u, f) \int_{T^3 \times \mathbb{R}^3} \left| u - \xi \right|^2 f \, dx \, d\xi + \psi_m \int_{T^3 \times \mathbb{R}^3} \left| \xi - \xi_c \right|^2 f \, dx \, d\xi$$

(81)

where $C_1, C_2, C_3$ are positive constants for sufficiently small $\sigma, ||\rho||_{L^\infty}$, and $||u||_{L^\infty}$,

$$C_1(\mu, \sigma, \rho, u, f) := \frac{3}{4} \mu + \rho_c - C\rho_c \|\rho\|_{L^\infty} - C\sigma \|\rho\|_{L^\infty}$$

$$- C\|\rho\|_{L^\infty} - C\|\rho\|_{L^2} - C\|\rho\|_{L^\infty} \|u\|_{L^\infty},$$

$$C_2(\mu, \sigma, \rho, u, f) := C\sigma \left[1 - C\|\rho\|_{L^\infty} - C\|u\|_{L^\infty} - C\|\rho\|_{L^\infty} - C\|\rho\|_{L^\infty} \|u\|_{L^\infty} - C\|\rho\|_{L^\infty} - C\|f\|_{L^\infty} - C\mu \sigma$$

$$- C\sigma(\lambda + \mu) - C\sigma^{-1} \left( ||u||_{L^\infty}^2 + ||\rho||_{L^\infty} \|u\|_{L^\infty} + ||\rho||_{L^\infty} \right),$$

$$C_3(\mu, \sigma, \rho, u, f) := \frac{1}{2} - C\rho_c - C\|\rho\|_{L^\infty}.$$ 

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