ON THE BASIN OF ATTRACTIONS FOR THE UNIDIRECTIONALLY COUPLED KURAMOTO MODEL IN A RING∗

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Abstract. We present the long-time dynamics of unidirectionally coupled identical Kuramoto oscillators in a ring, when each oscillator is influenced sinusoidally by a single preassigned oscillator. In this situation, for a large system with \( N \geq 5 \), it is well known that the synchronized states and the splay-state are the only stable equilibria by Gershgorin’s theorem and linear stability analysis, whereas for low-dimensional systems with \( N = 2,3 \) the synchronized state is the unique stable equilibrium. We present nontrivial proper subsets of synchronized and splay-state basins with positive Lebesgue measure in \( N \)-phase space. For the threshold case \( N = 4 \), we show that the splay-state is nonlinearly unstable by explicit construction of perturbations converging toward the synchronized state asymptotically.

Key words. Kuramoto model, basin of attractors, ring topology, synchronization, splay-state, stability

AMS subject classifications. 92D25, 74A25, 76N10

DOI. 10.1137/110829416

1. Introduction. Synchronization of coupled nonlinear oscillators is a common collective phenomenon observed in biological systems such as a group of fireflies, neurons, and cardiac pacemaker cells [1, 5, 29, 36]. The mathematical study of the synchronization of sinusoidally coupled oscillators was first pioneered by Kuramoto [20, 21], and recently it has been a hot topic in many scientific disciplines such as computer science, nonlinear dynamics, statistical physics, and network theory [19, 30] owing to its biological and engineering applications. For a general reference on synchronization, we refer the reader to [1, 4, 30, 33, 34]. Among many Kuramoto-type phase models with different coupling (or network) topologies, our interest in this paper lies in the Kuramoto model for identical oscillators with a unidirectional ring topology. It is reasonable to guess that different asymptotic patterns for Kuramoto oscillators depend on different network topologies. For example, it is well known that for identical Kuramoto oscillators with mean-field coupling (all-to-all coupling), the only stable phase-locked state is the synchronized state (in short, sync), which denotes the collapse of all phases into a single phase. Hence, except for a measure zero set, almost all initial configurations converge to the sync asymptotically. Of course, it is not known which configurations belong to a measure zero set consisting of initial configurations leading to a nonsync state asymptotically. As network topologies are varied, different stable phase-locked states will emerge asymptotically, and hence it would be an interesting problem to classify all possible stable asymptotic states and initial configurations converging to a given stable equilibria. This kind of question comes down to the identification of the basin of attractions in dynamical systems theory. As far as the authors know, there is little in the literature addressing this

*Received by the editors April 4, 2011; accepted for publication (in revised form) July 23, 2012; published electronically October 11, 2012.

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issue for the locally coupled Kuramoto model. For the all-to-all coupling, see [10, 9, 16, 18, 19]. Recently, Wiley, Strogatz, and Girvan [35] addressed the question of “the size of the sync basin” for the locally coupled Kuramoto model with symmetric forward and backward \( k \)-neighbor coupling and investigated the size of the sync basin as \( k \to \left[ \frac{N}{2} \right] \) numerically and analytically.

In this paper, we consider a group of identical Kuramoto oscillators labeled as \( 1, \ldots, N \), with each \( i \)th oscillator coupled by the \((i+1)\)th oscillator sinusoidally, e.g., a chain of oscillators with nearest forward neighbor coupling, in which the last \( N \)th oscillator is coupled by the first oscillator. Here the \((i+1)\)th oscillator may not be the immediate front of the \( i \)th oscillator. This situation appears in many engineering applications and biological modeling of animal locations (e.g., gaits of \( n \)-legged animals [15], twining of plants [23], rings of semiconductor lasers [32], circular antenna arrays [12, 13], and animal locomotion [11, 24]). Several models for the aforementioned situations have been proposed and studied [14, 22, 37]. For unidirectional coupling, the total phase is not conserved along the dynamics owing to the lack of symmetry in coupling. This lack of conserved quantities causes considerable mathematical difficulty. For example, the \( \ell^2 \) - and \( \ell^\infty \) -estimates employed in [10, 9, 16, 18] based on Lyapunov functionals cannot be applied directly in the present form. Of course this lack of symmetry in the coupling makes the asymptotic dynamics of Kuramoto oscillators richer than that of the mean-field case, because there is room for the emergence of other stable phase-locked states other than sync.

The novelty of this paper is threefold. We first present proper subsets of basins of synchronized and splay-states explicitly. For a large system with \( N \geq 5 \), the sync and splay-state are the only stable phase-locked states (i.e., the system is bistable), hence it would be worthwhile to get analytical statements saying that a given initial configuration will converge to sync, splay-state, or another unstable mode (e.g., chaotic motion). Second, we discuss the instability of the splay-state for \( N = 4 \) by explicit construction of unstable perturbations. For a four-particle system which is a missing case of Rogge and Aeyels’s work [31], since the Jacobian matrix evaluated at the splay-state is a zero matrix, the usual linear stability arguments are not applicable. We instead employ nonlinear arguments to show that the splay-state for a four-oscillator system is unstable by the explicit construction of unstable perturbations. Third, we provide a phase-locked state whose small perturbations lead to the sync and splay-state using nonlinear perturbation arguments asymptotically.

The paper is organized as follows. In section 2, we briefly present our model system and summarize Rogge and Aeyels’s results on the classification of stable phase-locked states. In section 3, we present proper subsets of sync and splay-state basins. For the sync basin, our proposed proper subset is independent of the number of oscillators. In contrast, the proper subset of the splay-state basin is only applicable for \( N \geq 5 \). In a low-dimensional system \( N \leq 4 \), the splay-state itself does not belong to the given proper subset. In section 4, we present two instability results using nonlinear perturbation arguments. First we show that the splay-state for the four-oscillator system in which the phase-difference between interacting oscillators is exactly \( \frac{\pi}{2} \) is nonlinearly unstable. Second, we present a phase-locked state \((\alpha, \ldots, \alpha, \pi - \alpha)\) that lies in the intersection of the boundary of sync and splay-state basins. Finally, section 5 is devoted to the summary of our main results.

2. Preliminaries. In this section, we discuss our model system and briefly summarize Rogge and Aeyels’s result [31] on the dynamics of unidirectionally coupled Kuramoto oscillators in a ring.

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Consider a unidirectionally coupled $N$-oscillator system via Kuramoto’s sinusoidal coupling in a ring. We denote the $N$-oscillators by the point rotors with labels $1, \ldots, N$, and let $\theta_i = \theta_i(t) \in \mathbb{R}$ be the phase of the $i$th oscillator. The location of the $(i+1)$th oscillator does not need to be adjacent to the $i$th oscillator. Assume that the $i$th oscillator is influenced by the $(i+1)$th oscillator only, and the $N$th oscillator is influenced by the first oscillator. In this circumstance, our governing system for $\theta_i$ reads as

$$\dot{\theta}_i = \Omega_i + K \sin(\theta_{i+1} - \theta_i), \quad \dot{\theta}_i := \frac{d\theta_i}{dt}, \quad i = 1, \ldots, N,$$

where $\theta_{N+1} \equiv \theta_1$, $\Omega_i$ is the natural frequency, and $K$ is a positive coupling strength.

In the following, we consider only identical oscillators with $\Omega_i = \Omega$, $i = 1, \ldots, N$. If necessary, we use a corotating reference frame $\theta_i \to \theta_i + \Omega t$ to assume

$$\Omega_i = \Omega_j = 0;$$

i.e., the time evolution of $\theta = (\theta_1, \ldots, \theta_N) \in \mathbb{R}^N$ is governed by the following system:

$$\dot{\theta}_i = K \sin(\theta_{i+1} - \theta_i), \quad t > 0. \quad (2.1)$$

Note that if $\theta(t)$ is the solution to the system (2.1), then its translation $\theta(t) + a$, $a \in \mathbb{Z}^N$, is also a solution, so that system (2.1) induces a dynamical system on $N$-tori $\mathbb{T}^N$.

As noticed in [31], it is convenient to work with the phase differences $\varphi_i \in [0, 2\pi)$:

$$\varphi_i \equiv \begin{cases} 
\theta_{i+1} - \theta_i \pmod{2\pi}, & i = 1, \ldots, N-1, \\
\theta_1 - \theta_N \pmod{2\pi}, & i = N,
\end{cases} \quad (2.2)$$

instead of the phase itself. Note that the phase difference $\varphi_i$ is taken from a value in the interval $[0, 2\pi)$ via modulo $2\pi$, and it follows from (2.2) that

$$\sum_{i=1}^{N} \varphi_i \equiv 0 \pmod{2\pi}.$$ 

Since $\varphi_i \in [0, 2\pi)$, this implies

$$\sum_{i=1}^{N} \varphi_i = 2k\pi \quad \text{for some } k \in \{0, 1, \ldots, N-1\}, \quad (2.3)$$

where the choice of $k$ depends on the configuration $\theta$.

To clarify the representation of $\varphi_i$, we consider the following three specific cases for $N = 3$:

- **Case 1** $(\theta_1, \theta_2, \theta_3) = (\theta_0, \theta_0, \theta_0)$, $\theta_0 \in [0, 2\pi)$: In this case,

$$\theta_2 - \theta_1 = 0, \quad \theta_3 - \theta_2 = 0, \quad \theta_1 - \theta_3 = 0.$$

Hence we have

$$(\varphi_1, \varphi_2, \varphi_3) = (0, 0, 0) \quad \text{and} \quad \sum_{i=1}^{3} \varphi_i = 0 \times 2\pi.$$ 

- **Case 2** $(\theta_1, \theta_2, \theta_3) = (0, \frac{2\pi}{3}, \frac{4\pi}{3})$: In this case,

$$\theta_2 - \theta_1 = \frac{2\pi}{3}, \quad \theta_3 - \theta_2 = \frac{2\pi}{3}, \quad \theta_1 - \theta_3 = -\frac{4\pi}{3}.$$
Since \(-\frac{4\pi}{3} \equiv \frac{2\pi}{3} \pmod{2\pi}\), we have
\[
(\varphi_1, \varphi_2, \varphi_3) = \left(\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}\right) \quad \text{and} \quad \sum_{i=1}^{3} \varphi_i = 1 \times 2\pi.
\]

- **Case 3** \((\theta_1, \theta_2, \theta_3) = (0, \frac{4\pi}{3}, \frac{2\pi}{3})\): In this case,
\[
\theta_2 - \theta_1 = \frac{4\pi}{3}, \quad \theta_3 - \theta_2 = -\frac{2\pi}{3}, \quad \theta_1 - \theta_3 = -\frac{2\pi}{3}.
\]
Then we have
\[
(\varphi_1, \varphi_2, \varphi_3) = \left(\frac{4\pi}{3}, \frac{4\pi}{3}, \frac{4\pi}{3}\right) \quad \text{and} \quad \sum_{i=1}^{3} \varphi_i = 2 \times 2\pi.
\]

On the other hand, the phase-difference \(\varphi_i\) satisfies
\[
\dot{\varphi}_i = K(\sin \varphi_{i+1} - \sin \varphi_i), \quad 1 \leq i \leq N, \quad \varphi_{N+1} = \varphi_1.
\]

In the following, we set
\[
\varphi := (\varphi_1, \ldots, \varphi_N).
\]

We first recall the definitions of two distinguished phase-locked solutions to system (2.1).

**Definition 2.1.** Let \(\theta\) and \(\varphi\) be the solutions to systems (2.1) and (2.4), respectively.

1. \(\theta\) is a synchronized state (sync) to (2.1) if and only if \(\varphi\) satisfies
   \[
   \varphi = 0_N.
   \]

2. \(\theta\) is the (geometric) splay-state if and only if \(\varphi\) satisfies
   \[
   \varphi = \frac{2k\pi}{N}1_N, \quad 1 \leq k \leq N - 1.
   \]

Here \(0_N\) and \(1_N\) denote constant vectors in \(\mathbb{T}^N\) defined by
\[
0_N := (0, \ldots, 0), \quad 1_N := (1, \ldots, 1).
\]

**Remark 2.1.**
1. The splay-state is a uniformly distributed traveling wave solution for system (2.1):
   \[
   \dot{\theta}_i = K \sin \frac{2k\pi}{N}, \quad \text{i.e.,} \quad \theta_i^L(t) = \theta_{i0} + K \left(\sin \frac{2k\pi}{N}\right)t.
   \]

2. The existence and stability issues for other Kuramoto-type models have been studied in [2, 3, 14, 25].

Since the phase-locked solution \(\varphi_i\) is an equilibrium solution to the system (2.4), it satisfies
\[
\sin \varphi_{i+1} - \sin \varphi_i = 0, \quad \text{or} \quad \varphi_{i+1} = \varphi_i, \quad \varphi_{i+1} = \pi - \varphi_i, \quad i = 1, \ldots, N.
\]
If we set one of the phase differences to $\alpha$, then
\[ \varphi_i \in \{\alpha, \pi - \alpha\}, \quad 1 \leq i \leq N. \]

We next summarize the result of Rogge and Aeyels [31] and Monzón and Paganini [28] concerning the existence and stability of equilibria to (2.4). In [31], the local stability of equilibria of (2.4) was shown by using Gershgorin’s theorem and standard linearization technique. In [28], combining certain Lyapunov function with LaSalle’s invariance principle, the authors showed that all trajectories $\varphi(t)$ converge to one of the equilibria of (2.4) asymptotically.

**Proposition 2.2** (see [28, 31]). Let $\varphi$ be the solution to system (2.4). Then we have the following:

1. Every equilibrium $\varphi := (\varphi_1, \varphi_2, \ldots, \varphi_N)$ to system (2.4) corresponds to a permutation of the vector
\[ (\alpha, \ldots, \alpha, \pi - \alpha, \ldots, \pi - \alpha), \]

satisfying
\[ m\alpha + (N-m)(\pi - \alpha) = 2k\pi, \quad 0 \leq k \leq N - 1, \]

where $\alpha \in S^1$, $1 \leq m \leq N$.

2. All trajectories $\varphi(t)$ converge to one of the equilibria of (2.4) asymptotically.

3. The equilibrium $\varphi = (\varphi_1, \ldots, \varphi_N)$ satisfying
\[ \varphi_i \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right) \quad \text{for all } i \]

is asymptotically stable.

4. The equilibrium $\varphi = (\varphi_1, \ldots, \varphi_N)$ satisfying
\[ \varphi_i \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \quad \text{for some } i \]

is unstable.

**Proof.** We refer the reader to [31] for statements (1), (3), and (4) and to [28] for statement (2), respectively.

**Remark 2.2.** 1. The equilibrium solution $\varphi_e$ to system (2.4) corresponds to the fixed point ($\alpha = 0$) or the traveling solution for $\theta$ with velocity $K \sin \alpha$, $\alpha \neq 0$.

2. Note that statements (3) and (4) in Proposition 2.2 are basically a local stability analysis, and hence they tell us nothing about the basin of attractions.

3. Note that the sync corresponds to $\varphi_i = 0$ for all $i$, and hence it is stable by (3).

4. The splay-state with $N \geq 5$ which is characterized by $\varphi_i = \frac{2\pi k}{N}, \quad 1 \leq k < \frac{N}{2}$, does satisfy (3), and therefore it is stable. However, for $N = 2, 3$ it follows from (4) that the splay-state is unstable. For $N = 4$, this corresponds to the marginal case, where $\varphi_i = \frac{\pi}{2}$ does not belong to any conditions in (3) and (4). Therefore, the results in Proposition 2.2 cannot reveal the stability in this case. However we will show that the splay-state with $N = 4$ is unstable in section 4.1.

5. For the linear stability results of the phase-locked states to the Kuramoto model, we refer the reader to [6, 8, 7, 26, 27, 34].
### 3. Proper subsets of sync and splay-state basins

In this section, we present proper subsets for the basins of synchronized and splay-states. For this, we employ a kind of partial differential equation (PDE) technique such as the continuation argument to construct global estimates from local estimates and the bootstrapping argument to improve our estimates from rough ones to finer ones. As long as $K$ is positive, our results presented in this paper are valid. If necessary, by the rescaling argument, we can set $K = 1$ in (2.1) throughout the paper without loss of generality, i.e.,

\begin{align}
\dot{\theta}_i &= \sin(\theta_{i+1} - \theta_i), \quad i = 1, \ldots, N, \quad t > 0, \\
\theta_i(0) &= \theta_{i0}.
\end{align}

#### 3.1. Sync basin

As we have discussed in the introduction, the total phase is not conserved along the flow (2.1); hence, it may not be a good idea to introduce the macro-micro decomposition employed for the mean-field cases [9, 16]. We instead work with the original phase variable $\theta_i$ directly. For a given time-varying configuration $\theta(t) = (\theta_1(t), \ldots, \theta_N(t))$, we introduce time-varying extremal indices $M = M(t)$, $m = m(t)$ and the phase-diameter

\begin{align}
\theta_M(t) := \max_{1 \leq i \leq N} \theta_i(t), \quad \theta_m(t) := \min_{1 \leq i \leq N} \theta_i(t), \quad D(\theta(t)) := \theta_M(t) - \theta_m(t).
\end{align}

Note that as long as $\theta_i(t)$ stays in a subset of $(0, 2\pi)$ for all $t$, $\theta_i(t)$ is a continuous function of $t$ (in fact analytic function of $t$ as a solution of system (2.1)). However, when $\theta_i$ touches $2\pi$ at some time, then it will be identified as 0, and hence $\theta_i$ will experience a jump discontinuity at that moment. Fortunately, thanks to the arguments employed in the proof of Lemma 3.1, the solutions to (2.1) will be confined to the interval $(0, \pi)$, where initial phases are taken from the interval $(0, \pi)$. Therefore, our solution $\theta(t) = (\theta_1(t), \ldots, \theta_N(t))$ to system (2.1) is a continuous function of $t$. In this situation, the phase-diameter $D(\theta(t))$ is also well defined. Owing to the analyticity of $\theta_i(t)$, the number of crossings between oscillators is countable and finite in any finite-time interval; hence the extremal phases $\theta_M(t)$ and $\theta_m(t)$ are Lipschitz continuous, in particular piecewise $C^\infty$. We will show that the set

\[ B_{sy} := \{ \theta \in \mathbb{R}^N : \theta_i \in (0, \pi), \quad i = 1, \ldots, N \} \]

is inside the sync basin. For this, we use a kind of bootstrapping argument; namely, we first show that the set $B_{sy}$ is a positively invariant set under the flow (2.1), and then we show that once the initial configuration lies in this set, the configuration shrinks to the sync asymptotically.

**Lemma 3.1.** Let $\theta = (\theta_1, \ldots, \theta_N)$ be a smooth solution to system (3.1) with an initial configuration $\theta_0 \in B_{sy}$. Then $\theta_M(t)$ is monotone decreasing and $\theta_m(t)$ is monotone increasing in $t$ so that

\[ D(\theta(t)) \leq D(\theta_0), \quad t \geq 0. \]

**Proof.** Before we present the detailed proof, we outline the proof as follows. The proof consists of two steps. First, we show that a priori as long as the angle stays in the interval $(0, \pi)$ in the time interval $[0, T)$, the maximal angle is nonincreasing, whereas the minimal angle is nondecreasing. Second, we show that the maximal time interval with the aforementioned monotonicity property is in fact $T = \infty$. Hence, for the entire time interval, the phase-diameter does not increase.
Step 1: For any $T \in (0, \infty)$, we assume that
\begin{equation}
0 \leq \theta_m(t) \leq \theta_M(t) \leq \pi, \quad t \in [0, T).
\end{equation}

Case 1 (nonincreasing property of $\theta_M$): Since $\theta_M$ is Lipschitz continuous in a finite-time interval $(0, T)$, and nondifferentiable times are at most finite in the time interval $(0, T)$, there exists $t_0 = 0 \leq t_1 \leq \cdots \leq t_n = T$ such that $\theta_M$ is differentiable on $(t_{k-1}, t_k)$, $k = 1, \ldots, n$.

For the interval $(t_{k-1}, t_k)$, we set
\[ \theta_M(t) := \theta_i(t), \quad t \in (t_{k-1}, t_k). \]

Then we use the a priori assumption (3.2) to find
\[ \dot{\theta}_i(t) = \sin(\theta_{i+1} - \theta_i) \leq 0, \quad t \in (t_{k-1}, t_k). \]

Hence, we have
\[ \theta_M(t) \leq \theta_M(t_{k-1}), \quad t \in (t_{k-1}, t_k). \]

We now use the above estimate and the continuity of $\theta_M(t)$ to obtain
\[ \theta_M(t) \leq \theta_M(s), \quad 0 \leq s \leq t \leq T. \]

Case 2 (nondecreasing property of $\theta_m$): Using an argument similar to that in Case 1, we obtain
\[ \theta_m(t) \geq \theta_m(s), \quad 0 \leq s \leq t \leq T. \]

In summary, we have shown that as long as the phases $\theta_i$ stay inside a half-circle $(0, \pi)$, the maximal phase is nonincreasing and the minimal phase is nondecreasing in time $t$.

Step 2: We now employ a PDE technique called a “continuation argument” or “continuous induction” to prove that we can choose $T = \infty$ in Step 1. For this, we define a set $\mathcal{C}$ and its supremum $T_0$:
\[ \mathcal{C} := \{ T > 0 \mid 0 \leq \theta_m(t) \leq \theta_M(t) < \pi, \quad t \in [0, T) \}, \quad T_0 := \sup \mathcal{C}. \]

By the assumption on the initial configuration and continuity of $\theta_i$, there exists $\varepsilon \ll 1$ such that
\[ 0 < \theta_i(t) < \pi, \quad t \in [0, \varepsilon), \]

i.e., $\varepsilon \in \mathcal{C}$. We next claim that
\[ T_0 = \infty. \]

Suppose not, i.e., $T_0 < \infty$. Then by definition of $T_0$, this yields
\begin{equation}
\lim_{t \to T_0^-} \theta_M(t) = \pi.
\end{equation}

Since $0 \leq \theta_m(t) \leq \theta_M(t) \leq \pi, t \in [0, T_0)$, we can apply the result of Step 1 and get
\[ \theta_m(0) \leq \theta_m(t) \leq \theta_M(t) \leq \theta_M(0), \quad t \in [0, T_0). \]
This implies
\[
\lim_{t \to T_0^-} \theta_M(t) \leq \theta_M(0) < \pi,
\]
which is contradictory to (3.3). Thus \(0 \leq \theta_m(t) \leq \theta_M(t) \leq \pi, t \geq 0\). By applying Step 1, we conclude that \(\theta_M\) is monotonically decreasing and \(\theta_m\) is monotonically increasing in \(t \geq 0\). \(\square\)

**Remark 3.1.** Consider a general system with time-dependent connection topology:

\[
\dot{\theta}_i = \sum_{j=1}^{N} c_{ji}(t) \sin(\theta_j - \theta_i),
\]

with the connection topology \(C(t) = (c_{ji}(t))\) satisfying the following two conditions:

(i) \(c_{ji}(t) = c_{ij}(t), 1 \leq i, j \leq N\);

(ii) for any two oscillators \(k\) and \(l\), there exists a sequence of connecting paths such that \(l = i_0 \rightarrow i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_n = k\), \(c_{i_1,i_{j+1}}(t) > 0, j = 0, \ldots, n - 1\).

For system (3.4) with time-independent connected topologies \(C = (c_{ij})\) satisfying the above two conditions, Lemma 3.1 was proved in [17] using the same arguments. The same arguments in [17] can be applied for system (3.4) as well.

In the following theorem, we refine the estimate in Lemma 3.1 to show that the phase-diameter \(D(\theta(t))\) is in fact decreased to zero asymptotically.

**Theorem 3.2.** Let \(\theta\) be the smooth solution to system (2.1) with an initial configuration \(\theta_0 \in B_{sy}\). Then we have

\[
\lim_{t \to \infty} \theta(t) = \theta_\infty 1_N,
\]

where \(\theta_\infty \in [\theta_m \theta_M]\) is a constant.

**Proof.** It follows from Lemma 3.1 that the phases are locked in the interval \((0, \pi)\), i.e.,

\[
0 < \theta_m(0) \leq \theta_i(t) \leq \theta_M(0) < \pi, \quad t \geq 0, \quad 1 \leq i \leq N.
\]

To conclude the desired convergence, it is convenient to consider the phase difference \(\varphi_i = \theta_{i+1} - \theta_i\) and claim

\[
\lim_{t \to \infty} \varphi_i(t) = 0.
\]

It follows from Proposition 2.2 that we have

\[
\lim_{t \to \infty} \varphi_i(t) = \alpha, \quad \pi - \alpha.
\]

- **Case 1:** Suppose
  \[
  \alpha \neq 0, \quad \pi.
  \]

Then it follows from (2.1) that

\[
\lim_{t \to \infty} \dot{\theta}_i(t) = \lim_{t \to \infty} \sin \varphi_i(t) = \sin \alpha \neq 0, \quad 1 \leq i \leq N.
\]

This contradicts (3.5).
Case 2: Suppose
\[ \alpha = \pi. \]
This is equivalent to
\[ \lim_{t \to \infty} (\theta_{i+1}(t) - \theta_i(t)) = \pi, \]
which also contradicts (3.5). Hence, \( \varphi_i \) must converge to 0, which means the formation of sync.

Remark 3.2. Since the dynamics (2.1) is invariant under a uniform phase shift, in fact, Lemma 3.1 and Theorem 3.2 also hold under the same initial configuration when the interval \((0, \pi)\) is replaced by \((\alpha, \alpha + \pi)\) for any \( \alpha \in \mathbb{R} \). Therefore the proper subset of the sync basin becomes
\[ B_{sy} := \{ \theta \in \mathbb{R}^N : \theta_i \in (\alpha, \alpha + \pi) \ \forall \ i \}. \]

3.2. Splay-state basin. In this part, we present a proper subset of a splay-state basin for a large system \( N \geq 5 \). For this, we are better off working with the phase difference \( \varphi_i \).

Recall that
\[ \dot{\varphi}_i = \sin \varphi_{i+1} - \sin \varphi_i, \quad 1 \leq i \leq N, \]
where the number of oscillators is assumed to satisfy \( N \geq 5 \). This restriction is necessary because in lower dimensions, say for \( N = 2, 3 \), the splay-state is asymptotically unstable, and for the critical case \( N = 4 \), we will show that the splay-state is also asymptotically unstable in the next section. Hence the result in this section is valid only for large dimensions \( N \geq 5 \).

As in the previous section, we set
\[ \varphi_M(t) := \max_{1 \leq i \leq N} \varphi_i(t) \quad \text{and} \quad \varphi_m(t) := \min_{1 \leq i \leq N} \varphi_i(t). \]

The extremal indices \( M \) and \( m \) depend on \( t \). As \( \theta_M \) and \( \theta_m \), the extremal phase differences \( \varphi_M \) and \( \varphi_m \) are also Lipschitz continuous and piecewise differentiable. We now define a set:
\[ B_{sp} := \{ \varphi = (\varphi_1, \ldots, \varphi_N) \in \mathbb{R}^N : 0 \leq \varphi_m < \varphi_M < \frac{\pi}{2} \}. \]

Lemma 3.3. Let \( \varphi \) be the smooth solution to system (3.6) with an initial condition \( \varphi_0 \in B_{sp} \). Then \( \varphi_M \) is nonincreasing and \( \varphi_m \) is nondecreasing in \( t > 0 \).

Proof. We employ the same argument as Lemma 3.1.

• (Step 1): Suppose for any \( T \in (0, \infty) \) we have
\[ 0 \leq \varphi_m(t) \leq \varphi_M(t) \leq \frac{\pi}{2}, \quad t \in [0, T). \]

Then in this interval \([0, T)\), for any \( 1 \leq i \leq N \), we have
\[ 0 \leq \frac{\varphi_{i+1} + \varphi_i}{2} \leq \frac{\pi}{2}, \quad -\frac{\pi}{4} \leq \frac{\varphi_{M+1} - \varphi_M}{2} \leq 0, \quad 0 \leq \frac{\varphi_m - \varphi_{m+1}}{2} \leq \frac{\pi}{4}. \]
and hence we have
\[
\dot{\varphi}_M(t) = \sin \varphi_{M+1} - \sin \varphi_M = 2 \cos \left( \frac{\varphi_{M+1} + \varphi_M}{2} \right) \sin \left( \frac{\varphi_{M+1} - \varphi_M}{2} \right) \leq 0, \quad \text{a.e. } t \in [0, T),
\]
\[
\dot{\varphi}_m(t) = \sin \varphi_{m+1} - \sin \varphi_m = 2 \cos \left( \frac{\varphi_{m+1} + \varphi_m}{2} \right) \sin \left( \frac{\varphi_{m+1} - \varphi_m}{2} \right) \geq 0.
\]
Therefore the smooth function \( \varphi_M \) is nonincreasing and \( \varphi_m \) is nondecreasing in the time interval \([0, T)\).

- (Step 2): We define a set \( C \) and \( T_0 \):
\[
C := \left\{ T > 0 \mid 0 \leq \varphi_m \leq \varphi_M < \frac{\pi}{2}, \quad t \in [0, T) \right\}, \quad T_0 := \sup C.
\]
Since \( \varphi_M(0) < \frac{\pi}{2} \), by the continuity of \( \varphi_M \), there exists \( \delta_1 > 0 \) such that
\[
\varphi_M(t) < \frac{\pi}{2}, \quad t \in [0, \delta_1).
\]
Since \( 0 \leq \varphi_m(0) \leq \varphi_M(0) < \frac{\pi}{2} \), we have
\[
\dot{\varphi}_m(0) = \sin \varphi_{m+1}(0) - \sin \varphi_m(0) = 2 \cos \left( \frac{\varphi_{m+1}(0) + \varphi_m(0)}{2} \right) \sin \left( \frac{\varphi_{m+1}(0) - \varphi_m(0)}{2} \right) \geq 0.
\]
We now claim that there exists \( \delta_2 > 0 \) such that
\[
\varphi_m(t) \geq 0, \quad t \in [0, \delta_2).
\]

The proof of claim (3.9). It is enough to consider the case \( \dot{\varphi}_m(0) = 0 \). In this case, we have
\[
\varphi_{m+1}(0) = \varphi_m(0).
\]
Since \( \varphi_m(0) < \varphi_M(0) \), there exists \( k \geq 2 \) such that
\[
\varphi_{m+k}(0) = \varphi_{m+k-1}(0) = \varphi_{m+k-2}(0) = \cdots = \varphi_m(0).
\]
By direct calculation via induction, we obtain
\[
\begin{align*}
\dot{\varphi}_{m+k-1}(0) & > 0; \\
\dot{\varphi}_{m+k-2}(0) & = 0, \quad \varphi_{m+k-2}(0) = \dot{\varphi}_{m+k-1}(0) \cos \varphi_{m+k-1}(0) > 0; \\
\text{for each } 3 \leq i \leq k, \quad \varphi_{m+i-1}(0) & = 0, \quad 1 \leq s \leq i - 1, \quad \varphi_{m+i-1}(0) > 0.
\end{align*}
\]
Hence, we get
\[
\varphi_{m}^{(s)}(0) = 0, \quad 1 \leq s \leq k - 1, \quad \varphi_{m}^{(k)}(0) > 0,
\]
which yields the claim (3.9).

By (3.8) and (3.9), we have \( \min \{ \delta_1, \delta_2 \} \in C \); i.e., the definition of \( T_0 \) makes sense.
We next claim that 

\[ T_0 = \infty. \]

Suppose not, i.e., \( T_0 < \infty \). Then by the definition of \( T_0 \), this yields

\[ \lim_{t \to T_0^-} \varphi_M(t) = \frac{\pi}{2}. \tag{3.10} \]

Since \( 0 \leq \varphi_m(t) \leq \varphi_M(t) \leq \frac{\pi}{2}, t \in [0, T_0) \), we can apply the result of Step 1 and get

\[ \varphi_m(0) \leq \varphi_m(t) \leq \varphi_M(t) \leq \varphi_M(0), \quad t \in [0, T_0). \]

This implies

\[ \lim_{t \to T_0^-} \varphi_M(t) \leq \varphi_M(0) < \frac{\pi}{2}, \]

which gives a contradiction to (3.10). Thus \( 0 \leq \varphi_m(t) \leq \varphi_M(t) \leq \frac{\pi}{2}, t \geq 0 \). By applying Step 1, we conclude that \( \varphi_M \) is nonincreasing and \( \varphi_m \) is nondecreasing in \( t \geq 0 \).

Remark 3.3. In fact, we can refine the arguments employed in Lemma 3.3 to get refined estimates for the strict monotonicity of \( \varphi_M \) and \( \varphi_m \):

\[ \varphi_M \text{ is strictly decreasing and } \varphi_m \text{ is strictly increasing.} \]

This will be proved of Step 1 in Theorem 3.2.

Theorem 3.4. Let \( \varphi = \varphi(t) \) be a smooth solution to system (3.6) with the initial configuration \( \varphi_0 \in B_{sp} \) and

\[ \sum_{i=1}^{N} \varphi_i(0) = 2k\pi \quad \text{for some } k \text{ with } 1 \leq k < \frac{N}{4}. \]

Then \( \varphi(t) \) converges to the splay-state, i.e.,

\[ \lim_{t \to \infty} \varphi(t) = \frac{2k\pi}{N} 1_N. \]

Proof.

- Step 1: We refine the result of Lemma 3.3 as follows:
  (i) \( \varphi_m \) is strictly increasing;
  (ii) \( \varphi_M \) is strictly decreasing.

The proof of the claim: (i) Note that

\[ \dot{\varphi}_m = \sin \varphi_{m+1} - \sin \varphi_m \geq 0. \]

Suppose \( \varphi_m \) is not strictly increasing, i.e., we have

\[ \sin \varphi_{m+1} = \sin \varphi_m \quad \text{on some open interval } I. \tag{3.11} \]

It follows from Lemma 3.3 that we have

\[ 0 \leq \varphi_i < \frac{\pi}{2} \quad \text{for all } i. \]
Then relation (3.11) implies that

$$\phi_{m+1} = \phi_m \quad \text{in} \quad I.$$  

This also yields

$$K(\sin \phi_{m+2} - \sin \phi_{m+1}) = \dot{\phi}_{m+1} = \dot{\phi}_m = K(\sin \phi_{m+1} - \sin \phi_m) \quad \text{in} \quad I.$$

Hence, we have

$$\sin \phi_{m+2} = \sin \phi_{m+1} \quad \text{in} \quad I.$$  

We apply the above argument continuously to obtain

$$\phi_l = \phi_m \quad \text{in} \quad I$$  

for all $$1 \leq l, m \leq N,$$ which implies $$\phi$$ is an equilibrium state in $$I$$. Therefore, $$\varphi(t)$$ is the equilibrium solution for all $$t \geq 0$$. However, this contradicts the initial condition which is not an equilibrium initially. Thus $$\varphi_m$$ is strictly increasing.

(ii) For the second part, we basically use the same argument to conclude the desired result.

- **Step 2:** It follows from Step 1 that $$\varphi_m$$ and $$\varphi_M$$ are strictly increasing and decreasing, respectively. Hence, we have

$$0 \leq \varphi_m(0) < \varphi_m(t) < \varphi_M(t) < \varphi_M(0) < \pi/2.$$

We now take limit $$t \to \infty$$ and use the strict monotonicity of $$\varphi_m$$ and $$\varphi_M$$ to find

$$0 \leq \varphi_m(0) < \lim_{t \to \infty} \varphi_m(t) \leq \lim_{t \to \infty} \varphi_M(t) < \varphi_M(0) < \pi/2.$$  

It follows from Proposition 2.2 that

$$\lim_{t \to \infty} \varphi_m(t), \lim_{t \to \infty} \varphi_M(t) \in \{\alpha, \pi - \alpha\}.$$  

Since

$$\alpha \in \left(0, \frac{\pi}{2}\right) \iff \pi - \alpha \in \left(\frac{\pi}{2}, \pi\right),$$

(3.12) and (3.13) imply

$$\lim_{t \to \infty} \varphi_i(t) = \alpha \in \left(0, \frac{\pi}{2}\right), \quad 1 \leq i \leq N.$$  

Note that it follows from (3.6) that we have a conservation law:

$$\sum_{i=1}^{N} \varphi_i(t) = \sum_{i=1}^{N} \varphi_i(0) = 2k\pi, \quad t \geq 0.$$  

Thus, we have

$$N\alpha = \lim_{t \to \infty} \sum_{i=1}^{N} \varphi_i(t) = 2k\pi \quad \text{or} \quad \alpha = \frac{2k\pi}{N}.$$  

Therefore, $$\varphi(t)$$ must converge to the splay-state $$\frac{2k\pi}{N}1$$ as $$t \to \infty$$.  

**Remark 3.4.** In the initial condition of Theorem 3.4, the existence of an integer $$k$$ satisfying $$1 \leq k < \frac{N}{4}$$ is automatically guaranteed by $$N \geq 5$$. Hence Theorem 3.4 is applicable only for large systems with $$N \geq 5$$. As we discussed in the introduction, for $$N = 2$$ and 3, the splay-state is not stable.
4. Nonlinear instability of phase-locked states. In this section, we present two instability results for two phase-locked states. For notational simplicity, we set

\[
\mathcal{E}(m, N - m; \alpha) := (\alpha, \ldots, \alpha, \pi - \alpha, \ldots, \pi - \alpha).
\]

The first instability result is about the splay-state \( \mathcal{E}(4, 0, \frac{\pi}{2}) \). Our second result treats some phase-locked states \( \mathcal{E}(N - 1, 1, \frac{\pi}{N - 2}) \) lying on the boundary of sync and splay-state basins.

4.1. Instability of \( \mathcal{E}(4, 0, \frac{\pi}{2}) \). Consider a one-parameter family of perturbations of the splay-state \( \mathcal{E}(4, 0, \frac{\pi}{2}) \):

\[
P_0(\delta) := \left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right) - \delta\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{2}\right), \quad \delta \in [0, \pi).
\]

We define an ordered set:

\[
\mathcal{S}_0 := \left\{ \varphi : 0 < \pi - \varphi_4 \leq \varphi_3 \leq \varphi_2 \leq \varphi_1 \leq \frac{\pi}{2} - \frac{\delta}{6} \right\}.
\]

We next briefly outline our strategy to verify the instability of the splay-state \( \mathcal{E}(4, 0, \frac{\pi}{2}) \). Our strategy can be divided into two steps.

- **Step 1** (instantaneous emergence of an ordering): There exists \( t_0 = t_0(\delta) \) such that
  \[
  \varphi(0) = P_0(\delta) \quad \Rightarrow \quad \varphi(t) \in \mathcal{S}_0, \quad t \in (0, t_0).
  \]

- **Step 2** (finite-time evolution toward the set \( B_{sy} \)):\n  \[
  T_0 := \sup\{T > 0 | \varphi(t) \in \mathcal{S}_0, \quad t \in [0, T]\} < \infty, \quad \varphi_4(T_0) = \pi.
  \]

Once Step 2 is done, by the continuity of \( \varphi_i \), we can show that there exists \( T^* > T_0 \) such that

\[
\theta(T^*) \in B_{sy}.
\]

Hence, we can now apply Theorem 3.1 for a new data \( \theta(T^*) \) to get the desired estimate.

**Lemma 4.1.** Let \( \varphi = \varphi(t) \) be a solution to system (3.6) with initial data \( \varphi(0) = P_0(\delta) \). Then there exists \( t_0 = t_0(\delta) > 0 \) such that

\[
\varphi(t) \in \mathcal{S}_0, \quad t \in (0, t_0).
\]

**Proof.** Note that \( \dot{\varphi}_i, \ddot{\varphi}_i, \) and \( \dddot{\varphi}_i \) satisfy

\[
\dot{\varphi}_i = \sin \varphi_{i+1} - \sin \varphi_i,
\]

\[
\ddot{\varphi}_i = \dot{\varphi}_{i+1} \cos \varphi_{i+1} - \dot{\varphi}_i \cos \varphi_i,
\]

\[
\dddot{\varphi}_i = \ddot{\varphi}_{i+1} \cos \varphi_{i+1} - (\ddot{\varphi}_i) \sin \varphi_{i+1} - \dot{\varphi}_i \cos \varphi_i + (\dot{\varphi}_i)^2 \sin \varphi_i.
\]

- **Case 1** (\( \varphi_1(t) < \frac{\pi}{2} - \frac{\delta}{6} \)): By direct calculation, it is easy to see that

\[
\varphi_1(0) = \dot{\varphi}_2(0) = 0 \quad \implies \quad \ddot{\varphi}_1(0) = 0,
\]

\[
\ddot{\varphi}_3(0) < 0 \quad \implies \quad \ddot{\varphi}_2(0) = K \dot{\varphi}_3(0) \cos \varphi_3(0) < 0 \quad \implies \quad \dddot{\varphi}_1(0) = K \dot{\varphi}_2(0) \cos \varphi_2(0) < 0.
\]
Thus it follows from Taylor’s theorem that
\[ \varphi_1(t) = \varphi_1(0) + \frac{\varphi_1'(0)}{6} t^3 + \mathcal{O}(t^4), \quad t \ll 1. \]
Hence, we have
\[ \varphi_1(t) < \varphi_1(0), \quad t \in (0, \tau_1) \text{ for some } \tau_1 > 0. \]

- **Case 2** (\( \varphi_2(t) < \varphi_1(t) \)): Since \( \varphi_2'(0) = \varphi_1'(0) \) and \( \varphi_2''(0) < \varphi_1''(0) \), we have \( \varphi_2'(t) < \varphi_1(t), \quad t \in (0, \tau_2) \text{ for some } \tau_2 > 0. \)

This and the assumption \( \varphi_1(0) = \varphi_2(0) \) imply
\[ \varphi_2(t) < \varphi_1(t), \quad t \in (0, \tau_2). \]

- **Case 3** (\( \varphi_3(t) < \varphi_2(t) \)): By the same argument as for Case 2, we use \( \varphi_3'(0) = \varphi_2'(0) \) and \( \varphi_3''(0) < \varphi_2''(0) \) to find
\[ \varphi_3(t) < \varphi_2(t), \quad t \in (0, \tau_3) \text{ for some } \tau_3 > 0. \]

- **Case 4** (\( 0 < \pi - \varphi_4(t) < \varphi_3(t) \)): By direct calculation, we have
\[ 0 < \pi - \varphi_4(0) = \pi - \frac{\delta}{2} < \frac{\pi}{2} - \frac{\delta}{6} = \varphi_3(0). \]

By the continuity of \( \varphi_3 \) and \( \varphi_4 \), we have
\[ 0 < \pi - \varphi_4(t) < \varphi_3(t), \quad t \in (0, \tau_4) \text{ for some } \tau_4. \]

Finally, we set
\[ t_0 := \min_{1 \leq i \leq 4} \tau_i \]
to get the desired result.

**Theorem 4.2.** Let \( \varphi = \varphi(t) \) be a solution to system (3.6) with initial data \( \varphi(0) = P(\delta) \). Then we have
\[ \lim_{t \to \infty} \varphi(t) = 0_N. \]
This implies that the splay-state \( \mathcal{E}(4,0,\frac{\pi}{2}) \) is nonlinearly unstable.

**Proof.** First, it follows from Lemma 4.1 that
\[ T_0 := \sup \{ T > 0 \mid \varphi(t) \in S_0, \quad t \in [0,T) \} \]
does make sense. Note that if
\[ 0 < \pi - \varphi_4(t) \leq \varphi_3(t) \leq \varphi_2(t) \leq \varphi_1(t) \leq \frac{\pi}{2} - \frac{\delta}{6}, \quad t > 0, \]
then we have
\[ 0 \geq \varphi_1 = \sin \varphi_2 - \sin \varphi_1 \geq \sin(\pi - \varphi_4) - \sin \varphi_1 = -\varphi_4, \quad t > 0. \]
Our proof consists of three steps.

- **Step 1:** We first claim that
  \[ T_0 < \infty. \]
  Suppose not, i.e., \( T_* = \infty \). Then we have
  \[ 0 < \varphi_4(t) \leq \pi \quad \text{for all } t \geq 0. \]
  Furthermore, since \( 0 \leq \dot{\varphi}_4(t) \leq 2 \) for all \( t \geq 0 \) by (4.2), we have
  \[ 0 = \lim_{t \to \infty} \dot{\varphi}_4(t) = \lim_{t \to \infty} (\sin \varphi_1(t) - \sin(\pi - \varphi_4(t))). \]
  Since \( 0 \leq \pi - \varphi_4(t) \leq \varphi_1(t) < \frac{\pi}{2}, t \geq 0 \), we have
  \[ (4.3) \quad \lim_{t \to \infty} \varphi_1(t) = \lim_{t \to \infty} (\pi - \varphi_4(t)). \]
  We integrate relation (4.2) from 0 to \( t \) to find
  \[ (4.4) \quad \varphi_4(0) - \varphi_4(t) \leq \varphi_1(t) - \varphi_1(0). \]
  We take \( t \to \infty \) to get
  \[ \varphi_4(0) + \varphi_1(0) \leq \lim_{t \to \infty} \varphi_1(t) + \lim_{t \to \infty} \varphi_4(t). \]
  The above inequality and (4.3) imply
  \[ 0 = \lim_{t \to \infty} \varphi_1(t) - \lim_{t \to \infty} (\pi - \varphi_4(t)) = -\pi + \lim_{t \to \infty} \varphi_1(t) + \lim_{t \to \infty} \varphi_4(t) \]
  \[ \geq -\pi + \varphi_1(0) + \varphi_4(0) = \frac{\delta}{3} > 0, \]
  which is contradictory. Thus \( T_0 < \infty. \)

- **Step 2:** We next show that
  \[ \varphi_4(T_0) = \pi. \]
  Suppose not, i.e., \( \varphi_4(T_0) < \pi \). Then it follows from (4.4) that
  \[ \varphi_1(T_0) + \varphi_4(T_0) \geq \varphi_1(0) + \varphi_4(0) = \pi + \frac{\delta}{3}. \]
  Moreover, we have
  \[ 0 < \pi - \varphi_4(T_0) \leq \varphi_1(T_0) \leq \varphi_1(0) < \frac{\pi}{2} < \varphi_4(T_0) < \pi. \]
  This yields
  \[ \dot{\varphi}_4(T_0) = 2 \sin \left( \frac{\varphi_1(T_0) - \varphi_4(T_0)}{2} \right) \cos \left( \frac{\varphi_1(T_0) + \varphi_4(T_0)}{2} \right) > 0, \]
  where we used the fact that
  \[ \varphi_1(T_0) - \varphi_4(T_0) < 0, \quad \varphi_1(T_0) + \varphi_4(T_0) > \pi. \]
  We next show that \( \dot{\varphi}_k(T_0) < 0, \ k = 1, 2, 3. \) The proof for this is done inductively.
First note that
\begin{equation}
\dot{\varphi}_3(t) = \sin \varphi_4(t) - \sin \varphi_3(t) \leq 0, \quad t \in [0, T_0].
\end{equation}

If \( \dot{\varphi}_3(T_0) = 0 \), then we have
\[ \ddot{\varphi}_3(T_0) = \dot{\varphi}_4(T_0) \cos \varphi_4(T_0) < 0. \]

Therefore, we have \( \dot{\varphi}_3(T_0 - \eta) > 0 \) for sufficiently small \( \eta > 0 \), which is contradictory to (4.5). Hence we have
\[ \dot{\varphi}_3(T_0) < 0. \]

Similarly, we have
\begin{equation}
\dot{\varphi}_3(T_0) < 0 \implies \dot{\varphi}_2(T_0) < 0 \implies \dot{\varphi}_1(T_0) < 0.
\end{equation}

Finally, by the continuity of \( \varphi_4 \) and \( \dot{\varphi}_k \) for \( k = 1, 2, 3 \), there exists \( T_* > T_0 \) such that
\[ \varphi_4(t) < \pi, \]
\[ \sin \varphi_{k+1}(t) - \sin \varphi_k(t) = \dot{\varphi}_k(t) < 0, \quad k = 1, 2, 3, \quad t \in [T_0, T_*], \]

which implies
\[ 0 < \pi - \varphi_4(t) < \varphi_3(t) < \varphi_2(t) < \varphi_1(t) \leq \frac{\pi}{2} - \frac{\delta}{6}, \quad t \in [T_0, T_*]. \]

Hence, \( T_* \in \{ T > 0 \mid \varphi(t) \in S_0, \ t \in [0, T) \} \), which gives a contradiction to the definition of \( T_0 \). Therefore we have
\[ \varphi_4(T_0) = \pi. \]

**Step 3:** We complete the proof by applying Theorem 3.2.

Since
\[ 0 \leq \pi - \varphi_4 \leq \varphi_3 \leq \varphi_2 \leq \varphi_1 < \frac{\pi}{2}, \quad t \in (0, T_0], \]
we have
\[ \dot{\varphi}_3 = \sin(\pi - \varphi_4) - \sin \varphi_3 \geq \sin(\pi - \varphi_4) - \sin \varphi_1 = -\dot{\varphi}_4, \quad t \in (0, T_0]. \]

By integrating over \([0, T_0] \) and \( \varphi_4(T_0) = \pi \), we get
\[ \varphi_3(T_0) \geq \varphi_3(0) + \varphi_4(0) - \varpi = \frac{\delta}{3} > 0, \]

which yields \( 0 < \varphi_k(T_0) < \frac{\pi}{2}, \ k = 1, 2, 3 \). This and \( \dot{\varphi}_4(T_0) > 0 \) imply that there exists \( T^* > T_0 \) such that
\begin{equation}
\varphi_4(T^*) > \pi \quad \text{and} \quad 0 < \varphi_k(T^*) < \frac{\pi}{2}, \quad k = 1, 2, 3.
\end{equation}

Finally, using (4.7) and the conservation law
\[ \sum_{i=1}^{4} \varphi_i(T^*) = \sum_{i=1}^{4} \varphi_i(0) = 2\pi, \]
we have
\[ \theta_k(T^*) \in (\alpha, \alpha + \pi), \quad 1 \leq k \leq 4, \text{ for some } \alpha \in S^1. \]

Hence, we can apply Theorem 3.2 (see Remark 3.2) by starting again from \( T^* \).

Remark 4.1. For the repulsive coupling strength \( K < 0 \), we can easily show the instability of \( \mathcal{E}(4, 0; \frac{\pi}{2}) \). Using an argument similar to that used in the proof of Theorem 4.2, we may consider the initial condition
\[ \begin{align*}
\varphi_1(0) & = \varphi_2(0) = \varphi_3(0) = \frac{\pi}{2} + \frac{\delta}{6}, \\
\varphi_4(0) & = \frac{\pi}{2} - \frac{\delta}{2}.
\end{align*} \]

Then similarly we define
\[ S_0 := \left\{ \varphi \mid \frac{\pi}{2} + \frac{\delta}{6} \leq \varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \pi - \varphi_4 < \pi, \quad t \in (0, T] \right\}, \]

to complete the proof by obtaining \( \varphi_4(t) = 0 \) in finite time \( t \).

4.2. Instability of \( \mathcal{E}(N - 1, 1; \alpha) \). In this part, we show that for \( N \geq 5 \) the phase-locked state \( \mathcal{E}(N - 1, 1; \frac{\pi}{N - 2}) \) of (3.6) belongs to the intersection of the boundaries between sync basin and splay-state basin. By Proposition 2.2 (4) for \( N \geq 5 \), the phase-locked state \( \mathcal{E}(N - 1, 1; \frac{\pi}{N - 2}) \) of (3.6) is nonlinearly unstable; thus it might be a candidate for the configuration we are looking for. In the following, we restrict our discussion to the case \( N \geq 5 \).

4.2.1. From the perturbation \( \mathcal{E}(N - 1, 1; \alpha) \) to the splay-state. Since the estimates given in the following two subsections are rather technical and long, we next present our strategy. Recall that our purpose is to show that there exist perturbations of the phase-locked state \( \mathcal{E}(N - 1, 1; \frac{\pi}{N - 2}) \) leading to the sync and splay-state, respectively. For the perturbations leading to splay-states, we introduce a one-parameter family of perturbations:
\[ P^+_1(\alpha, \varepsilon) := (\alpha, \ldots, \alpha, \pi - \alpha) + (0, \ldots, 0, \varepsilon, -\varepsilon), \]
\[ \alpha := \frac{(2k - 1)\pi}{N - 2} \text{ for some } 1 \leq k < \frac{N}{4}, \quad \varepsilon < \frac{\pi}{2} - \alpha. \]

We define an ordered set:
\[ S^+_1 := \left\{ \varphi : 0 < \alpha \leq \varphi_1 \leq \varphi_2 \leq \cdots \leq \varphi_{N-1} \leq \pi - \varphi_N < \frac{\pi}{2} \right\}. \]

The idea proving the convergence to the splay-state is similar to the arguments employed in section 4.1 to verify that the splay-state with four oscillators is unstable. Our strategy again can be divided into two steps, for a given initial configuration \( \varphi_0 = P^+_1(\alpha, \varepsilon) \).

- **Step 1** (instantaneous emergence of an ordering): There exists \( t_0 = t_0(N, \varepsilon) \) such that
  \[ \varphi_0 = P^+_1(\alpha, \varepsilon) \implies \varphi(t) \in S^+_1, \quad t \in (0, t_0). \]

Note that \( \varphi_0 = P^+_1(\alpha, \varepsilon) \) implies \( \sum_{i=1}^{N} \varphi_i(0) = 2k\pi \).

- **Step 2** (finite-time evolution toward the set \( \mathcal{B}_{sp} \)):
  \[ T_0 := \sup\{ T > 0 \mid \varphi(t) \in S^+_1, \quad t \in [0, T) \} < \infty, \quad \varphi_N(T_0) = \frac{\pi}{2}. \]
Once Step 2 is done, by the continuity of \( \varphi_i \), we can show that there exists \( T^* > T_0 \) such that

\[
\varphi(T^*) \in B_{sp}.
\]

Hence we can now apply Theorem 3.2 for a new data \( \varphi(T^*) \) to get the desired estimate.

**Lemma 4.3.** Let \( \varphi = \varphi(t) \) be the smooth solution to system (3.6) with an initial configuration \( \varphi_0 = P_1^+(\alpha, \varepsilon) \). Then there exists a positive constant \( t_0 = t_0(N, \varepsilon) \) such that

\[
\varphi(t) \in S^+_1, \quad 0 < t \leq t_0.
\]

**Proof.** First, we obtain the following relations via induction:

\[
\varphi_N(0) < 0;
\]

(4.8) \[
\varphi_{N-1}(0) = 0, \quad \varphi_{N-1}(0) > 0, \quad \varphi_{N-2}(0) > 0;
\]

for each \( 3 \leq k \leq N - 1 \), \( \varphi^{(s)}_{N-k}(0) = 0 \), \( 1 \leq s \leq k - 2 \), \( \varphi^{(k-1)}_{N-k}(0) > 0 \).

- **Case 1** (\( \varphi_1(t) > 0, \pi - \varphi_N(t) < \frac{\pi}{2}, \varphi_{N-1}(t) > \varphi_{N-2}(t) \)): Since

\[
\varphi_1(0) = \alpha > 0, \quad \pi - \varphi_N(0) = \alpha + \varepsilon < \frac{\pi}{2}, \quad \varphi_{N-1}(0) - \varphi_{N-2}(0) = \varepsilon > 0,
\]

by continuity, we have

\[
\varphi_1(t) > 0, \quad \pi - \varphi_N(t) < \frac{\pi}{2}, \quad \varphi_{N-1}(t) > \varphi_{N-2}(t), \quad t \in (0, \tau] \quad \text{for some } \tau.
\]

- **Case 2** (\( \pi - \varphi_N(t) > \varphi_{N-1}(t) \)): By using (4.8), we get

\[
\pi - \varphi_N(0) - \varphi_{N-1}(0) = 0, \quad \frac{d}{dt} \bigg|_{t=0} (\pi - \varphi_N - \varphi_{N-1}) > 0,
\]

which implies

\[
\pi - \varphi_N(t) > \varphi_{N-1}(t), \quad t \in (0, \tau_1] \quad \text{for some } \tau_1 \leq \tau.
\]

- **Case 3** (\( \varphi_{N-k}(t) > \varphi_{N-k-1}(t) \)): For each \( 2 \leq k \leq N - 2 \), by (4.8), we have

\[
\varphi^{(k-2)}_{N-k}(0) - \varphi^{(k-2)}_{N-k-1}(0) = 0 \quad \text{and} \quad \varphi^{(k-1)}_{N-k}(0) - \varphi^{(k-1)}_{N-k-1}(0) > 0,
\]

which implies

\[
\varphi^{(k-2)}_{N-k}(t) - \varphi^{(k-2)}_{N-k-1}(t) > 0, \quad t \in (0, \tau_k] \quad \text{for some } \tau_k \leq \tau_1.
\]

By (4.8), if \( \varphi^{(k-3)}_{N-k}(0) - \varphi^{(k-3)}_{N-k-1}(0) = 0 \), then we have

\[
\varphi^{(k-3)}_{N-k}(t) - \varphi^{(k-3)}_{N-k-1}(t) > 0, \quad t \in (0, \tau_k].
\]

If we continue this argument inductively, we have

\[
\varphi_{N-k}(t) - \varphi_{N-k-1}(t) > 0, \quad 0 < t \leq t_0 \quad \text{for } t_0 = \min\{\tau_k \mid 2 \leq k \leq N - 2\}.
\]
THEOREM 4.4. Under the same assumption as Lemma 4.3, then we have
\[ \lim_{t \to \infty} \varphi(t) = \frac{2k\pi}{N} 1_N. \]

**Proof.** We set
\[ T_0 := \sup \{ T > 0 \mid \varphi(t) \in S^+_1, t \in [0, T] \}. \]

Then it follows from Lemma 4.3 that \( T_0 \) is well defined. Note that if
\[ \alpha \leq \varphi_1(t) \leq \varphi_2(t) \leq \cdots \leq \varphi_{N-1}(t) \leq \pi - \varphi_N(t) < \frac{\pi}{2}, \quad t > 0, \]
then we have
\[ (4.9) \quad 0 \leq \dot{\varphi}_1 = \sin \varphi_2 - \sin \varphi_1 \leq \sin(\pi - \varphi_N) - \sin \varphi_1 = -\dot{\varphi}_N, \quad t > 0. \]

Our proof consists of two steps.

- **Step 1:** We first show that \( T_0 < \infty \).

Suppose not, i.e., \( T_0 = \infty \). Then \( \varphi_N(t) \geq \frac{\pi}{2} \) for all \( t \geq 0 \), i.e., \( \varphi_N(t) \) is bounded below. Furthermore, since \( -2 \leq \dot{\varphi}_N(t) \leq 0 \) for all \( t \geq 0 \) by (4.9), we have
\[ 0 = \lim_{t \to \infty} \dot{\varphi}_N(t) = \lim_{t \to \infty} (\sin \varphi_1(t) - \sin(\pi - \varphi_N(t))). \]

Since \( 0 < \varphi_1(t) \leq \pi - \varphi_N(t) < \frac{\pi}{2} \) for \( t \geq 0 \), we get
\[ (4.10) \quad \lim_{t \to \infty} \varphi_1(t) = \lim_{t \to \infty} (\pi - \varphi_N(t)). \]

We integrate relation (4.9) from 0 to \( t \) to find
\[ (4.11) \quad \varphi_N(0) - \varphi_N(t) \geq \varphi_1(t) - \varphi_1(0). \]

We take \( t \to \infty \) to get
\[ \varphi_1(0) + \varphi_N(0) \geq \lim_{t \to \infty} \varphi_1(t) + \lim_{t \to \infty} \varphi_N(t). \]

The above inequality and (4.10) imply
\[ 0 > -\varepsilon = \varphi_1(0) + \varphi_N(0) - \pi \geq \lim_{t \to \infty} \varphi_1(t) - \lim_{t \to \infty} (\pi - \varphi_N(t)) = 0, \]
which gives a contradiction. Hence \( T_0 < \infty \).

We next claim that \( \varphi_N(T_0) = \frac{\pi}{2} \). Suppose not, i.e., \( \varphi_N(T_0) < \frac{\pi}{2} \).

Then we have
\[ \alpha \leq \varphi_1(t) \leq \varphi_2(t) \leq \cdots \leq \varphi_{N-1}(t) \leq \pi - \varphi_N(t) < \frac{\pi}{2}, \quad t \in (0, T_0]. \]

Then we use (4.11) to get
\[ \varphi_1(T_0) + \varphi_N(T_0) \leq \varphi_1(0) + \varphi_N(0) = \pi - \varepsilon. \]
Moreover, since
\[ 0 < \varphi_1(0) \leq \varphi_1(T_0) \leq \pi - \varphi_N(T_0) \leq \frac{\pi}{2} < \varphi_N(T_0) < \pi, \]
we get
\[ \dot{\varphi}_N(T_0) = 2 \sin \left( \frac{\varphi_1 - \varphi_N}{2} \right) \cos \left( \frac{\varphi_1 + \varphi_N}{2} \right) \bigg|_{t=T_0} < 0. \]

We next show that
\[ \dot{\varphi}_k(T_0) > 0, \quad 1 \leq k \leq N - 1. \tag{4.12} \]

First note that
\[ \dot{\varphi}_{N-1}(t) = \sin \varphi_N(t) - \sin \varphi_{N-1}(t) \geq 0, \quad t \in [0, T_0]. \tag{4.13} \]

If \( \dot{\varphi}_{N-1}(T_0) = 0 \), then we have
\[ \dot{\varphi}_{N-1}(T_0) = \dot{\varphi}_N(T_0) \cos \varphi_N(T_0) > 0. \]

Therefore, we have \( \dot{\varphi}_{N-1}(T_0 - \eta) < 0 \) for sufficiently small \( \eta > 0 \), which is contradictory to (4.13). Hence we have
\[ \dot{\varphi}_{N-1}(T_0) > 0. \]

Similarly, we can induce the following inductively:
\[ \dot{\varphi}_k(T_0) > 0 \implies \dot{\varphi}_{k-1}(T_0) > 0, \quad 2 \leq k \leq N - 1. \]

Finally, by (4.12) and \( \varphi_N(T_0) > \frac{\pi}{2} \), there exists \( T_0 \) with \( T_* > T_0 \) such that
\[ \varphi_{N}(t) > \frac{\pi}{2}, \quad \varphi_1(t), \varphi_2(t), \ldots, \varphi_{N-1}(t) > 0, \quad t \in [T_0, T_*]. \]

Thus, we have
\[ \alpha \leq \varphi_1(t) \leq \varphi_2(t) \leq \cdots \leq \varphi_{N-1}(t) \leq \pi - \varphi_N(t) < \frac{\pi}{2}, \quad t \in [T_0, T_*]. \]

Hence, \( T_* \in \{ T > 0 \mid \varphi(t) \in S^+_N, \ t \in [0, T] \} \), which gives a contradiction to the definition of \( T_0 \), and therefore we have
\[ \varphi_N(T_0) = \frac{\pi}{2}. \]

\[ \bullet \text{Step 2: We end the proof by applying Theorem 3.4.} \]

Since
\[ \alpha \leq \varphi_1(t) \leq \varphi_2(t) \leq \cdots \leq \varphi_{N-1}(t) \leq \pi - \varphi_N(t) \leq \frac{\pi}{2}, \quad t \in (0, T_0], \tag{4.14} \]
we have
\[ \varphi_{N-1} = \sin(\pi - \varphi_N) - \sin \varphi_{N-1} \leq \sin(\pi - \varphi_N) - \sin \varphi_1 = -\varphi_N, \quad t \in (0, T_0]. \]

In particular, since it follows from the proof of Lemma 4.3 that
\[ \alpha \leq \varphi_1(t) < \varphi_{N-1}(t) < \pi - \varphi_N(t) < \frac{\pi}{2}, \quad t \in (0, t_0] \quad \text{for some} \ t_0, \]
we have $\varphi_{N-1} < -\varphi_N$, $t \in (0, t_0]$. Thus, by integrating over $[0, T_0]$ and $\varphi_N(T_0) = \frac{\pi}{2}$, we get

$$\varphi_{N-1}(T_*) < \varphi_{N-1}(0) + \varphi_N(0) - \frac{\pi}{2} = \frac{\pi}{2}. $$

This and (4.14) imply $\varphi_M(T_0) = \varphi_N(T_0)$, and thus $\varphi_M(T_0) < 0$.

Therefore, by the continuity of $\varphi$, and (4.14), there exists $T^* > T_0$ such that

$$0 \leq \varphi_m(T^*) \leq \varphi_M(T^*) < \frac{\pi}{2}. $$

Finally, by using the conservation law

$$\sum_{i=1}^{N} \varphi_i(T^*) = \sum_{i=1}^{N} \varphi_i(0) = 2k\pi, $$

we can apply Theorem 3.4 by starting again from $T^*$.  

**4.2.2. From the perturbation of $E(N-1; 1; \alpha)$ to the sync.** In this part, we present a family of perturbations of the phase-locked state $E(N-1; 1; \frac{\pi}{N-2})$ converging to the sync $E(N; 0; 0)$ asymptotically. We basically use the same argument as in the previous case. For the perturbations leading to the sync, we introduce a one-parameter family of perturbations:

$$P_1^{-}(\alpha, \varepsilon) := (\alpha, \ldots, \alpha, \pi - \alpha) + (0, \ldots, 0, -\varepsilon, \varepsilon), \quad \alpha := \frac{\pi}{N-2}, \quad \varepsilon < \alpha. $$

We define an ordered set:

$$S_1^- := \left\{ \varphi : 0 < \pi - \varphi_N \leq \varphi_{N-1} \leq \cdots \leq \varphi_2 \leq \varphi_1 \leq \alpha \right\}. $$

The idea of proving the convergence to the sync is similar to the arguments employed in the previous subsection.

**Lemma 4.5.** Let $\varphi = \varphi(t)$ be the smooth solution to system (3.6) with an initial configuration $\varphi_0 = P_1^{-}(\alpha, \varepsilon)$. Then there exists a positive constant $t_0^{-} = t_0(N, \varepsilon)$ such that

$$\varphi(t) \in S_1^- , \quad 0 < t \leq t_0. $$

**Proof.** By direct calculation via induction, we obtain

$$\varphi_N(0) > 0;$$

(4.15) $\varphi_{N-1}(0) = 0, \quad \varphi_{N-1}(0) < 0, \quad \varphi_{N-2}(0) < 0;$$

for each $3 \leq k \leq N - 1$, $\varphi_{N-k}^{(s)}(0) = 0, \quad 1 \leq s \leq k - 2, \quad \varphi_{N-k}^{(k-1)}(0) < 0$.

- **Case 1** ($\varphi_1(t) < \alpha, \pi - \varphi_N(t) > 0, \varphi_{N-1}(t) < \varphi_{N-2}(t)$): Since

$$\varphi_{1}^{(N-2)}(0) < 0, \quad \pi - \varphi_N(0) = \alpha - \varepsilon > 0, \quad \varphi_{N-2}(0) - \varphi_{N-1}(0) = \varepsilon > 0,$$

by continuity, we have

$$\varphi_1(t) < \alpha, \quad \pi - \varphi_N(t) > 0, \quad \varphi_{N-2}(t) > \varphi_{N-1}(t), \quad t \in (0, \tau] \text{ for some } \tau.$$
Then it follows from Lemma 4.5 that

\[
\pi - \varphi_N(t) < \varphi_{N-1}(t), \quad t \in (0, \tau_1]
\]

for some \(\tau_1 \leq \tau\).

**Case 3** \(\varphi_{N-k}(t) < \varphi_{N-k-1}(t)\): For each \(2 \leq k \leq N-2\), by (4.8), we have

\[
\varphi_{N-k}(0) - \varphi_{N-k-1}(0) = 0 \quad \text{and} \quad \varphi_{N-k}(0) - \varphi_{N-k-1}(0) < 0,
\]

which implies

\[
\varphi_{N-k}(t) - \varphi_{N-k-1}(t) < 0, \quad t \in (0, \tau_k] \quad \text{for some} \ \tau_k \leq \tau_1.
\]

By (4.15), if \(\varphi_{N-k}(0) - \varphi_{N-k-1}(0) = 0\), then we have

\[
\varphi_{N-k}(t) - \varphi_{N-k-1}(t) < 0, \quad t \in (0, \tau_k].
\]

If we continue this argument inductively, we have

\[
\varphi_{N-k}(t) - \varphi_{N-k-1}(t) < 0, \quad 0 < t \leq t_0 \quad \text{for} \ t_0 = \min\{\tau_k \mid 2 \leq k \leq N-2\}.
\]

**Theorem 4.6.** Under the same assumption as that of Lemma 4.5, we have

\[
\lim_{t \to \infty} \varphi(t) = 0_N,
\]

i.e., the flow converges to the sync.

**Proof.** We set

\[
T_0 := \{T > 0 \mid \varphi(t) \in S_1^{-}, \ t \in [0, T)\}.
\]

Then it follows from Lemma 4.5 that \(T_0\) is well defined. Note that if

\[
0 < \pi - \varphi_N(t) \leq \varphi_{N-1}(t) \leq \cdots \leq \varphi_2(t) \leq \varphi_1(t) \leq \alpha, \quad t > 0,
\]

then we have

\[
0 \geq \varphi_1 = \sin \varphi_2 - \sin \varphi_1 \geq \sin(\pi - \varphi_N) - \sin \varphi_1 = -\varphi_N, \quad t > 0.
\]

Our proof consists of two steps.

- **Step 1:** We first show that

\[
T_0 < \infty.
\]

If we suppose \(T_0 = \infty\), then \(\varphi_N(t) \leq \pi\) for all \(t \geq 0\), i.e., \(\varphi_N(t)\) is bounded above.

Furthermore, since \(0 \leq \varphi_N(t) \leq 2\) for all \(t \geq 0\) by (4.16), we have

\[
0 = \lim_{t \to \infty} \varphi_N(t) = \lim_{t \to \infty} (\sin \varphi_1(t) - \sin(\pi - \varphi_N(t))).
\]

Since \(0 < \pi - \varphi_N(t) \leq \varphi_1(t) < \frac{\pi}{2}\) for \(t \geq 0\), we get

\[
\lim_{t \to \infty} \varphi_1(t) = \lim_{t \to \infty} (\pi - \varphi_N(t)).
\]
We integrate relation (4.16) from 0 to \( t \) to find
\[
(4.18) \quad \varphi_N(0) - \varphi_N(t) \leq \varphi_1(t) - \varphi_1(0).
\]
We take \( t \to \infty \) to get
\[
\varphi_1(0) + \varphi_N(0) \leq \lim_{t \to \infty} \varphi_1(t) + \lim_{t \to \infty} \varphi_N(t).
\]
The above inequality and (4.17) imply
\[
\epsilon = \varphi_1(0) + \varphi_N(0) - \pi \leq \lim_{t \to \infty} \varphi_1(t) - \lim_{t \to \infty} (\pi - \varphi_N(t)) = 0,
\]
which gives a contradiction. Hence \( T_0 < \infty \).

We now claim that \( \varphi_4(T_0) = \pi \). Suppose not, i.e., \( \varphi_4(T_0) < \pi \).

Then we have
\[
0 < \pi - \varphi_N(t) \leq \varphi_{N-1}(t) \leq \cdots \leq \varphi_2(t) \leq \varphi_1(t) \leq \alpha, \quad t \in (0, T_0].
\]
Then we use (4.18) to get
\[
\varphi_1(T_0) + \varphi_N(T_0) \geq \varphi_1(0) + \varphi_N(0) = \pi + \epsilon.
\]
Moreover, since
\[
0 < \pi - \varphi_N(T_0) \leq \varphi_1(T_0) \leq \varphi_1(0) < \frac{\pi}{2} < \varphi_N(T_0) < \pi,
\]
we get
\[
\varphi_N(T_0) = 2 \sin \left( \frac{\varphi_1 - \varphi_N}{2} \right) \cos \left( \frac{\varphi_1 + \varphi_N}{2} \right) \bigg|_{t=T_0} > 0.
\]
We next show that
\[
(4.19) \quad \dot{\varphi}_k(T_0) < 0, \quad 1 \leq k \leq N - 1.
\]
First, note that
\[
(4.20) \quad \dot{\varphi}_{N-1}(t) = \sin \varphi_N(t) - \sin \varphi_{N-1}(t) \leq 0 \quad t \in [0, T_0].
\]
If \( \dot{\varphi}_{N-1}(T_0) = 0 \), then we have
\[
\dot{\varphi}_{N-1}(T_0) = \dot{\varphi}_N(T_0) \cos \varphi_N(T_0) < 0.
\]
Therefore, we have \( \dot{\varphi}_{N-1}(T_0 - \eta) > 0 \) for sufficiently small \( \eta > 0 \), which is contradictory to (4.20). Hence we have
\[
\dot{\varphi}_{N-1}(T_0) < 0.
\]
Similarly, we can induce the following inductively:
\[
\dot{\varphi}_k(T_0) < 0 \quad \implies \quad \dot{\varphi}_{k-1}(T_0) < 0, \quad 2 \leq k \leq N - 1.
\]
Finally, by (4.19) and \( \varphi_N(T_0) < \pi \), there exists \( T_* \) with \( T_* > T_0 \) such that
\[
\varphi_N(t) < \pi, \quad \varphi_1(t), \varphi_2(t), \ldots, \varphi_{N-1}(t) < 0, \quad t \in [T_0, T_*].
\]
Thus, we have
\[ 0 < \pi - \varphi_N(t) \leq \varphi_{N-1}(t) \leq \cdots < \varphi_2(t) \leq \varphi_1(t) \leq \alpha, \quad t \in [T_0, T_*]. \]

Hence, \( T_* \in \{ T > 0 \mid \varphi(t) \in S^-_R, \ t \in [0, T) \} \), which gives a contradiction to the definition of \( T_0 \), and therefore we have
\[ \varphi_N(T_0) = \pi. \]

- **Step 2**: We end the proof by applying Theorem 3.2. Since
\[ 0 < \pi - \varphi_N(t) \leq \varphi_{N-1}(t) \leq \cdots < \varphi_2(t) \leq \varphi_1(t) \leq \alpha, \quad t \in (0, T_0], \]
we have
\[ \dot{\varphi}_{N-1} = \sin(\pi - \varphi_N) - \sin \varphi_{N-1} \geq \sin(\pi - \varphi_N) - \sin \varphi_1 = -\dot{\varphi}_N, \quad t \in (0, T_0], \]

In particular, since it follows from the proof of Lemma 4.5 that
\[ 0 < \pi - \varphi_N(t) < \varphi_{N-1}(t) < \varphi_1(t) < \alpha, \quad t \in (0, t_0] \text{ for some } t_0, \]
we have \( \dot{\varphi}_{N-1} > -\varphi_N, \ t \in (0, t_0] \). Thus, by integrating over \([0, T_0]\) and \( \varphi_N(T_0) = \pi \), we get
\[ \varphi_{N-1}(T_0) > \varphi_{N-1}(0) + \varphi_N(0) - \pi = 0, \]

which yields \( 0 < \varphi_k(T_0) < \frac{\pi}{2}, \ 1 \leq k \leq N - 1 \). This and \( \varphi_N(T_0) > 0 \) imply that there exists \( T^* > T_0 \) such that
\[ \varphi_N(T^*) > \pi \quad \text{and} \quad 0 < \varphi_k(T^*) < \frac{\pi}{2}, \ 1 \leq k \leq N - 1. \]

Finally, using (4.21) and the conservation law
\[ \sum_{i=1}^{N} \varphi_i(T^*) = \sum_{i=1}^{N} \varphi_i(0) = 2\pi, \]
we have
\[ \theta_k(T^*) \in (\alpha, \alpha + \pi), \ 1 \leq k \leq N, \quad \text{for some } \alpha \in S^1. \]

Hence we can apply Theorem 3.2 (see Remark 3.2) by starting again from \( T^* \).

5. **Conclusion.** We provided proper subsets for the basins of sync and splay-state for the unidirectionally coupled Kuramoto model in a ring. Recall that proper subsets for sync and splay-state basins are
\[
\begin{align*}
B_{sy} & := \{ \theta \in \mathbb{R}^N : \theta_i \in (\alpha, \alpha + \pi) \ \forall \ i \}, \quad \alpha \in \mathbb{R}, \\
B_{sp} & := \{ \varphi \in \mathbb{R}^N : 0 \leq \varphi_m \leq \varphi_M < \frac{\pi}{2} \}.
\end{align*}
\]

In Lemmas 3.1 and 3.3, we showed that these sets are positively invariant sets along the flow (2.1). Hence, the flows hit the above sets, and then evolve to the sync or splay-state asymptotically. Note that for \( N \leq 4 \), set \( B_{sp} \) is empty. In low dimensions...
future work.

and

The extension of our approach to the case of nonidentical oscillators and symmetric
does not use any algebraic tools such as Gershgorin’s theorem or linearization tricks.
equilibria by a direct nonlinear approach. Our proposed approach is elementary and
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= 4, the instability of the splay-state has already been established [31]. However
for N = 4, the stability of the splay-state has not been treated in [31], because the lin-
ear stability analysis there is not applicable. However, we provided a one-parameter
family of perturbations of the splay-state with N = 4 leading to the sync asymptotically. The idea is simply to show that the perturbations hit set \( B_{sp} \) in a finite
time so that the flow will converge to the sync asymptotically. Therefore, we com-
pletely resolved the issue of stability on the splay-states for the system (2.1). In low
dimensions, \( N \leq 4 \), the splay-state is nonlinearly unstable, whereas in high dimen-
sions, \( N \geq 5 \), the splay-state is stable. As an application of Theorems 3.2 and 3.4,
we showed that the equilibria \( \mathcal{E}(N-1, 1: \pi) \) lie in the intersection of boundaries of
sync and splay-state basins by explicitly constructing perturbations of the equilib-
ria \( \mathcal{E}(N-1, 1: \frac{\pi}{N-2}) \). In summary, we provided a stability and instability theory of
equilibria by a direct nonlinear approach. Our proposed approach is elementary and
does not use any algebraic tools such as Gershgorin’s theorem or linearization tricks.
The extension of our approach to the case of nonidentical oscillators and symmetric
and k-nearest-neighbor couplings will be an interesting problem to be investigated in
future work.

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