Notes by Eric Katz

TROPICAL GEOMETRY

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1. Introduction

1.1. **Two applications.** Let us begin with two examples of questions where tropical geometry is useful.

Example 1.1. *Can we represent an untied trefoil knot as a genus* 1 *curve of degree* 5 *in* $\mathbb{RP}^3$*?*

Yes, by means of tropical geometry. It turns out that it can be be represented by a rational degree 5 *curve but not by curve of genus greater than* 1 *since such a curve must sit on a quadric surface in* $\mathbb{RP}^3$.

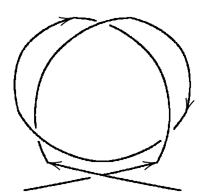


Figure 1. Untied trefoil.

Example 1.2. *Can we enumerate real and complex curves simultaneously by combinatorics? For example, there is a way to count curves in* $\mathbb{RP}^2$ *or* $\mathbb{CP}^2$ *through* $3d-1+g$ *points by using bipartite graphs.*

1.2. **Tropical Geometry.** Tropical geometry is algebraic geometry over the tropical semi-field, $(\mathbb{T}, \text{“}+\text{”}, \text{“}\cdot\text{”})$. The semi-field's underlying set is the half-open interval $[-\infty,\infty)$. The operations are given for $a, b \in \mathbb{T}$ by

$$\text{“}a+b\text{”} = \max(a,b) \quad \text{“}a\cdot b\text{”} = a+b.$$

The semi-field has the properties of a field except that additive inverses do not exist. Moreover, every element is an idempotent, “$a+a$” $= a$ so there is no way to adjoin inverses. In some sense algebra becomes harder, geometry becomes easier.

By the way, tropical geometry is named in honor of a Brazilian computer scientist, Imre Simon.

Two observations make tropical geometry easy. First, the tropical semiring $\mathbb{T}$ naturally has a Euclidean topology like $\mathbb{R}$ and $\mathbb{C}$. Second, the geometric structures are piecewise linear structures, and so tropical geometry reduces to a combination of combinatorics and linear algebra. The underlying geometric structure is known as an integer affine structure. Unfortunately, the underlying space will not be a manifold but rather a polyhedral complex. The tropical algebraic data will enrich the polyhedral complex with an integer affine structure.

In many cases, tropical objects are limits of classical objects (usually algebraic varieties) under certain degenerations. As an example, let us consider the classical objects of holomorphic curves in $\mathbb{CP}^N$, that is maps $f : S_g \to \mathbb{CP}^N$ from a Riemann surface to $\mathbb{CP}^N$ that satisfy that Cauchy-Riemann equations. The corresponding tropical objects are piecewise-linear graphs in $\mathbb{R}^N$ whose edges are weighted by integers that satisfy the following conditions: the edges have rational slopes; and the balancing condition described below is satisfied at vertices. These provide finitely many easily checked conditions. We should note that it is much more subtle to determine whether a tropical object is a limit of a classical object. We will describe some sufficient conditions in certain cases later.

The degeneration arises as follows. Look at the torus $(\mathbb{C}^*)^N \subset \mathbb{CP}^N$. There is a map

$$\begin{array}{cccc} \mathrm{Log} & : & (\mathbb{C}^*)^N & \to & \mathbb{R}^N \\ & & (z_1, \ldots, z_N) & \to & (\log|z_1|, \ldots, \log|z_N|). \end{array}$$

If $V = f(S_g)$ is the image of the holomorphic map, the set $\mathrm{Log}(V) \subset \mathbb{R}^N$ is called the amoeba of V.

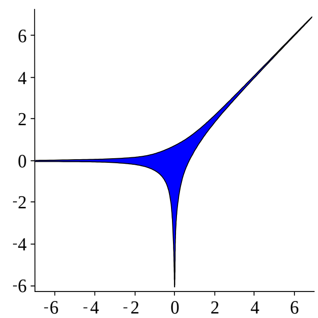


FIGURE 2. [22] The amoeba of the line $\{x + y + 1 = 0\} \subset (\mathbb{C}^*)^2$.

We may include Log in a family of maps,

$$\mathrm{Log}_t : (z_1, \ldots, z_N) \mapsto (\log_t |z_1|, \ldots, \log_t |z_n|)$$

As $t \mapsto \infty$, the amoeba will be dilated and the Hausdorff limit $\lim_{t\to\infty} \mathrm{Log}_t(V)$ is a union of rays at the origin. This limit contains homological information about the behavior near the ends but little else. Instead, consider a family of curves $V_t \subset (\mathbb{C}^*)^N$ parameterized by t. The limit $\lim_{t\to\infty} \mathrm{Log}_t(V)$ is a piecewise-linear object called a tropical curve.

Notice that when we apply Log, we lose information about the argument of points in V. We can therefore strengthen the correspondence between classical and tropical curves by including some information on phases in the tropical picture. These will give us phase-tropical curves. Using these data, under certain conditions, we may lift the tropical curve back to the classical curves.

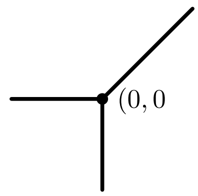


FIGURE 3. The limit of the amoeba of the line $\{x + y + 1 = 0\} \subset (\mathbb{C}^*)^2$.

1.3. **Organization.** The plan for the lectures is to do the following:

(1) Define tropical manifolds which will serve ambient spaces and on which we will discuss hypersurfaces, cycles, morphisms, and tropical equivalence.
(2) Outline the patchworking and non-Archimedian approaches to tropical geometry.
(3) Treat tropical curves in detail and provide the analogs of the Abel-Jacobi map and the Riemann-Roch theorem.
(4) Introduce phase-tropical curves and state the approximation theorem.

update list

1.4. **Ingredients in Tropical Geometry.** Here is a list of various influences on tropical geometry. This list is not exhaustive.

(1) Topological side - Viro's patchworking [28], Amoebas of Gelfand-Kapranov-Zelevinsky [16], non-Archimedean amoebas [11]
(2) Mirror Symmetry after Kontsevich-Soibelmann [18, 19] and Fukaya-Oh [12].
(3) Rigid analytic geometry after Berkovich [6]
(4) Study of valuations after Bergman [5] and Bieri and Groves [7]
(5) Toric Geometry following Khovanskii [17] and Fulton and Sturmfels [13]
(6) (p, q)-webs in physics by Aharony, Hanany and Kol [1].

The words "tropical" and "geometry" were put together in March of 2002 in Alta, Utah by Mikhalkin and Sturmfels. Initial interest was stimulated by a suggestion of Kontsevich that tropical varieties could be used to address enumerative questions.

1.5. **These notes.** These notes are based on a series of lectures given by the author at the University of Texas in February and March 2008. The notes are by Eric Katz. Several figures are from other papers of the author. Some figures are by Brian Katz. The note-taker takes responsibility for the other figures and for any inaccuracies in the notes.

The note-taker recommends as resources the following papers and books: [24, 21, 26, 23, 22, 20].

2. Tropical Polynomials

Let $\mathbb{T} = [-\infty, \infty)$ be the underlying space of the tropical semi-field. Let $\mathbb{T}^n = [-\infty, \infty)^n$ be its Cartesian power.

Definition 2.1. *A tropical polynomial is a finite sum*

$$\text{“}\sum a_J x^J\text{”}$$

for multi-indexes $J = (j_1, \dots, j_n) \in \mathbb{Z}^n$, $a_J \in \mathbb{T}$ *where*

$$x^J = \text{“}x_1^{j_1} \dots x_n^{j_n}\text{”}$$

Note that “” will be used to denote tropical operations while the classical operations will not be in quotes. Observe that

$$\text{“}\sum a_J x^J\text{”} = \max(a_J + j_1 x_1 + \cdots + j_n x_n).$$

The Laurent polynomial “x^{-1}” is not defined at $-\infty$ since “$(-\infty)^{-1}$” $\notin \mathbb{T}$. The Laurent polynomials that are defined on all of $\mathbb{T}^n$ are those for which $j_i \geq 0$, in other words, tropical polynomials. A tropical polynomial is convex on $(\mathbb{T}^*)^n = \mathbb{R}^n$.

Definition 2.2. *For* $U \subset \mathbb{T}^n$*, let* $\mathcal{F}$ *be the pre-sheaf where* $\mathcal{F}(U)$ *is the set of all functions on* U *that are restrictions of Laurent polynomials defined on* U*. The* structure sheaf*,* $\mathcal{O}$ *is the sheafification of* $\mathcal{F}$*. Elements of* $\mathcal{F}(U)$ *are called* regular functions*.*

Note that regular functions are convex. We also consider rational functions which are locally the difference of regular functions. If a rational function fails to be convex near a point, then it is certainly not regular there.

One can define spaces with tropical structures by locally modeling them on $\mathbb{T}^n$ and considering the semi-ringed space given by the sheaf of regular functions.

3. Integer Affine Structures

The space $\mathbb{T}^n$ is a manifold with corners. We can use $\mathbb{T}^n$ to model tropical spaces and give them an integer affine structure as an alternative to the semi-ringed space structure.

Definition 3.1. *An* integer affine linear transformation $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ *is a function that can be expressed as the composition of a* $\mathbb{Z}$*-linear transformation* $\mathbb{R}^n \to \mathbb{R}^m$ *followed by an arbitrary translation in* $\mathbb{R}^m$*.*

Definition 3.2. *An* integer affine structure *on a smooth* n*-dimensional manifold* M *is an open covering* $\{U_\alpha\}$ *of* M *with charts* $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ *such that the transition maps* $\phi_\beta \circ \phi_\alpha^{-1}$ *are the restriction of integer affine transformations* $\Phi_{\beta\alpha} : \mathbb{R}^n \to \mathbb{R}^n$*.*

Observe the lattice spanned by $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ in T_pU is preserved by integer affine transformations. In fact, an integer affine structure on a manifold is equivalent to a consistent family of lattices in the tangent spaces of points of M. This family of lattices can be obtained from the regular functions on U.

3.1. **Integer affine structures on the torus.** Let us consider a family of examples corresponding to abelian varieties. Let $\Lambda \cong \mathbb{Z}^N$ be a full-rank lattice in $\mathbb{R}^N$. Let $\mathbb{R}^N$ have the standard integer affine structure coming from the lattice $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$. This integer affine structure is preserved by translations by elements of Λ and therefore descends to the quotient $\mathbb{R}^N/\Lambda$.

For two different choices Λ, Λ', the quotients $\mathbb{R}^N/\Lambda$, $\mathbb{R}^N/\Lambda'$ may be different as integer affine manifolds. Consider the the standard square lattice Λ in $\mathbb{R}^2$ and some generic lattice Λ'. A curve $C \subset \mathbb{R}^N/\Lambda$ is said to have rational slope if it is tangent to an integer vector, say $\vec{v}$ at every point. Any such curve C lifts to a line in some integer vector direction in $\mathbb{R}^2$. Since $t\vec{v} \in \Lambda$ for some $t \in \mathbb{R}_+$ such curves close up in $\mathbb{R}^2/\Lambda$ while they do not in $\mathbb{R}^2/\Lambda'$.

3.2. **Projective space.** The fundamental example of a manifold with corners locally modeled on $\mathbb{T}^n$ is tropical projective space.

Definition 3.3. *Tropical projective space* $\mathbb{TP}^N = \mathbb{T}^N \setminus \{(-\infty)^N\}/\sim$ *where* $\sim$ *is the relation defined by*

$$(x_0, \ldots, x_N) \sim \text{“}\lambda \cdot (x_0, \ldots, x_N)\text{”}, \quad \lambda \in \mathbb{T}^*.$$

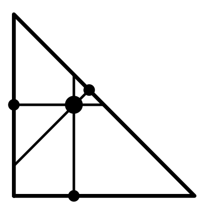
For example, $\mathbb{TP}^1$ is covered by two charts $U_i = \{x_i \neq -\infty\}$ for $i = 0, 1$. The map $f_0 : (x_0, x_1) \mapsto x_1 - x_0$ provides a chart from U_0 to $\mathbb{T}$, while the map $f_1 : (x_0, x_1) \mapsto x_0 - x_1$ provides a chart from U_1 to $\mathbb{T}$. The transition map is $x \mapsto -x$ or equivalently $x \mapsto$ “$\frac{1}{x}$”. As a topological space, $\mathbb{TP}^1$ is isomorphic to the closed interval, but its integer affine structure makes it looks like $[-\infty, \infty]$.

Analogously, $\mathbb{TP}^N$ is topologically a closed simplex, but geometrically, its interior is $\mathbb{R}^N$.

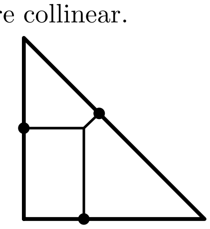
3.3. **Some geometry.** A tropical line in $\mathbb{TP}^2$ is the set where the max of a function “$ax_0 + bx_1 + cx_2$” (for $a, b, c \in \mathbb{T}$) is achieved twice.

Theorem 3.4. $\mathbb{TP}^2$ *satisfies the Fano plane axiom. Pick four points* (p_1, p_2, p_3, p_4) *in tropical general position. Draw all tropical lines through all pairs of points. Pairs of such lines intersect in the four given points plus three additional points. Those three points are collinear.*

This theorem does not hold classically except over fields of characteristic 2. We give an example of a Fano configuration. We let three of the four points be the vertices of the $\mathbb{TP}^2$. The fourth point is chosen generically in the interior. The lines between the pairs of the points turn out to be the edges of the simplex together with the three lines



The three intersection points are collinear.



3.4. **Manifolds of Finite Type.** Now, we would like to impose some additional conditions on tropical manifolds. We ask that our integer affine manifolds X to be Hausdorff and of finite-type.

Definition 3.5. *An integer affine manifold* X *is said to be finite type if there are finitely many charts* $\phi_\alpha : U_\alpha \to \mathbb{T}^n$ *such that* $\cup U_\alpha = X$ *and each chart* U_α *is extendable in the following sense: there exists an open set* $V_\alpha \supseteq U_\alpha$ *and* $\psi_\alpha : V_\alpha \to \mathbb{T}^n$ *such that* $\overline{\phi(U_\alpha)} \subseteq \psi_\alpha(V_\alpha)$.

Example 3.6. *The open interval* $(0,1)$ *with the standard integer affine structure is not of finite type, but* $\mathbb{R}$ *is. Here the two ends are modeled on the open end of* $[-\infty,\infty)$.

4. Hypersurfaces Associated to Regular Functions

In this section, we develop the theory analogous to the theory of principal divisors.

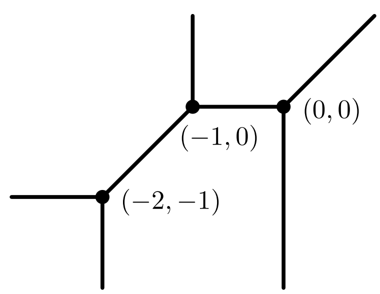
Let $U \subset X$ be a subset of a tropical manifolds. Let $f : U \to \mathbb{T}$ be a tropically regular function, that is the restriction of a Laurent polynomial to $U \subset \mathbb{T}^n$ so that it is everywhere defined.

Definition 4.1. *The* tropical hypersurface V_f *associated to* f *is the set of points* $x \in U$ *where* "$\frac{1}{f}$" *is not locally regular.*

Lemma 4.2. V_f *is the locus where* f *is strictly convex.*

Proof. The condition that "$\frac{1}{f}$" is not locally regular is equivalent to $-f$ not locally being the restriction of a tropical polynomial. But $-f$ is the minimum of some of linear functions with integer slope hence concave. Since tropical polynomials are convex, so for $-f$ to be the restriction of a tropical polynomial, it must be linear. It follows that "$\frac{1}{f}$" not being regular is equivalent to f not being linear, hence being strictly convex. □

Example 4.3. *The hypersurface of* "$1+3x+3x^2+2y+3xy''$ *in* $\mathbb{T}^2$ *is the following graph in the plane*



V_f is a polyhedral complex in $U \subset \mathbb{T}^n$. That is, it is a union of convex polyhedra of dimension $n-1$ such that the intersection of any two of them is a common face. Moreover every facet is contained in a hyperplane with rational slope. We see this by writing f as the maximum of some monomials,

$$f = \max(a_J + J \cdot x).$$

This function is strictly convex when the maximum is achieved by two different terms. The locus where this happens is cut out by linear equations and linear inequalities. Each facet comes of V_f comes with a natural weight by a positive integer. A facet F is defined by a set of monomials, $\{a_{J_k} + J_k \cdot x\}$ which are maximized on F. The slopes J_k lie along a line in $\mathbb{Z}^n$. The weight, $w(F)$ is the lattice length of this line. Alternatively, we may define the weight as

$$w(F) = \max |\gcd(J_{k_1} - J_{k_2})|$$

where the maximum ranges over pairs (k_1, k_2) and $J_{k_1} - J_{k_2}$ is considered as an n-tuple of integers.

In the case $n = 2$, the tropical curve V_f satisfies the following *balancing condition*. Let p be a vertex of V_f and $e_1, \ldots, e_l$ be the edges containing p. Let $v_1, \ldots, v_l$ be the primitive integer vectors along $e_1, \ldots, e_l$ pointing away from p. (Here primitive means the greatest common divisor of the components is 1). Then,

$$\sum w(e_i)v_i = 0.$$

For general n, the hypersurface V_f satisfies the balancing condition. For any $n-2$-dimensional face E of V_f, let $F_1, \ldots, F_l$ be the facets containing E. Let $\lambda : \mathbb{R}^n \to \mathbb{R}^2$ be the projection along E. Then $C = \lambda(\cup F_i)$ is a one- dimensional integral polyhedral complex in $\mathbb{R}^2$. Give the edges in C the weights coming form V_f. Then C is balanced.

5. Balanced Cycles and Tropical Intersection Theory

We may formulate the balancing condition abstractly. We recommend [24] and [2] for details. Let us first do this for 1-dimensional integral polyhedral complexes in $\mathbb{R}^n$. A balanced 1-dimensional weighted integral polyhedral complex is a graph in $\mathbb{R}^n$ whose edges are (possibly half-infinite) line segments with rational slope, weighted by integers. It is balanced if for every vertex p, $\sum w(e_i)v_i = 0$ where e_i are the edges adjacent to p, and v_i is a primitive integer vector along e_i pointing away from p.

We may extend this to k-dimensional weighted integral polyhedral complexes by saying that such a complex is balanced if for every codimension 1 face, E adjacent to facets $F_1, \ldots, F_l$, and projection $\lambda : \mathbb{R}^n \to \mathbb{R}^{n-k}$ along E, $\lambda(\cup F_i)$ is balanced. We call such complexes k-cycles. If all weights are positive, they are said to be effective.

Cycles can be deformed by parallel translations of linear subspaces as long as the combinatorics of adjacency is preserved as in Figure 4. The slopes cannot be deformed. If F is a top-dimensional cell in a cycle Z, then F lies in an affine

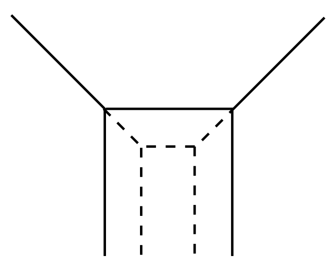


Figure 4. Deformation of a Tropical Cycle

subspace with integral slope. If we translate this subspace to contain the origin, it contains a lattice Λ_F. Two cycles Z, Z' of complementary dimensions k, k', $k+k' = n$ are said to intersect *tropically transversely* if for every point of intersection $p \in Z \cap Z'$ belongs to the relative interior of top-dimensional cells F, F' of Z, Z' so that F, F' lie in transverse subspaces. The multiplicity m_p of such an intersection is the product of the weights with a particular lattice index and is defined to be

$$m_p = w_Z(F)w_Z(F')[Z^n : \Lambda_F + \Lambda'_F].$$

The intersection number is the sum of the lattice indices

$$Z \cdot Z' = \sum_{p \in Z \cap Z'} m_p.$$

This intersection number is stable under deformations of Z and Z' [13, 27]. In fact, the balancing condition can be reformulated as saying that: Z is balanced if and only if its intersection number with a complementary rational subspace L is invariant under translations of L.

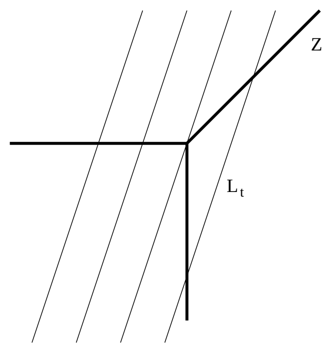


FIGURE 5. Family of Lines intersecting a cycle

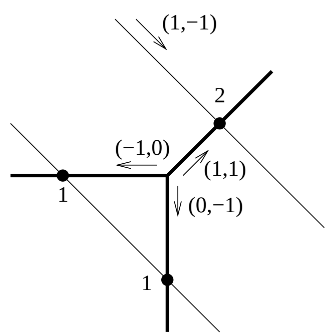


FIGURE 6. Invariance of Intersection Number

Example 5.1. *Consider the intersection of the tropical zero locus of "$x + y + 1$" with a rational line in the direction $(1, -1)$. If the divisor intersects the line in two points, the intersection number is*

$$\begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} = 2$$

while if it intersects in a single point, the intersection number is

$$\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = 2.$$

This is illustrated in Figure 6

We may also define projection π of a k-cycle Z along a linear subspace L of complementary dimension in such a way that the image, $\pi(Z)$ is a k-dimensional linear subspace L' orthogonal to L and such that the multiplicity of $\pi(Z)$ as a one-celled tropical variety is $Z \cdot L$.

One can similar define the intersection of more than 2 cycles and intersections when the dimensions are not complementary.

In $\mathbb{R}^n$, in the case that an intersection is not transverse, one can define it as a limit of perturbations making it transverse. This is the stable intersections in [27]. Here, for two cycles A, B in a manifold X, the intersection $A \cdot B$ will a union of faces of the expected dimension in $A \cap B$ with certain multiplicities. There may be situations where cycles have negative self-intersections.

There is a tropical analog of Bezout's theorem. A cycle $Z^k \subset \mathbb{TP}^n$ is said to be of degree $d \in \mathbb{Z}$ if d is the intersection number with an $(n-k)$-dimensional projective subspace. Recall that a projective $(n-k)$-space is a translate of the $(n-k)$-skeleton of the normal fan to a face of a standard n-simplex. For example, we may take our subspace to be a coordinate subspace in $\mathbb{TP}^n$.

Theorem 5.2 (Bezout's Theorem)**.** *If $Z_1, \ldots, Z_l$ are cycles of degrees $d_1, \ldots, d_l$ whose codimensions sum to n, then $Z_1 \cdot Z_2 \cdot \cdots \cdot Z_l = d_1 d_2 \ldots d_l$.*

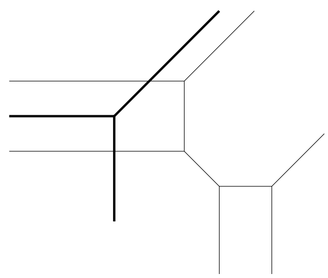


FIGURE 7. A tropical line intersecting a tropical conic

Bezout's theorem is illustrated in Figure 7 where it is shown that a line and a conic intersect in two points. If f is a tropical polynomial of degree d, then V_f is a cycle in $\mathbb{T}^n$ and $\overline{V_f}$ is a cycle of degree d in $\mathbb{TP}^n$.

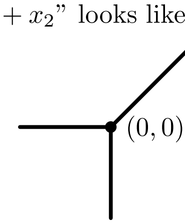
Let us explore the case of hyperplanes in detail. Hyperplanes are given by tropical linear forms

$$\text{“}a_0 + a_1x_1 + \cdots + a_nx_n\text{”}.$$

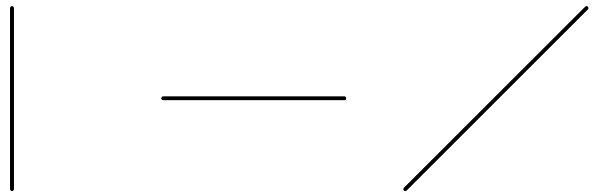
In the case where no a_i is $-\infty$, we may change coordinates by $y_i = x_i + a_i$ and scale to get

$$\text{”}0 + y_1 + \cdots + y_n\text{”}.$$

In that case, the tropical hyperplane is just a translate of a standard hyperplane. In $\mathbb{TP}^2$, the hyperplane “$0 + x_1 + x_2$” looks like



It may even degenerate to any of the following lines and their translates



It may even coincide with the coordinate axes with the boundary lines of $\mathbb{TP}^2$.

6. Patchworking

Tropical curves in the plane were described both as limit of amoebas of curves in the complex plane and as the tropical divisor of a tropical polynomial in two variables. We outline the proof of the slightly more general fact that every tropical hypersurface is a limit of amoebas of complex hypersurfaces. The proof uses the patchworking technique of Viro [28].

Let $f =$ "$\sum a_J x^J$" be a tropical polynomial. Let $F_t = \sum t^{a_J} x^J$ be the corresponding family of classical polynomials. Near any point $x \in \mathbb{R}^n$, the behavior of $\mathrm{Log}_t(V(F_t))$ is determined by the dominating monomials of f. We make this clear in the following example.

Example 6.1. *Let*

$$f(x, y) = \text{“}0 + x + y + (-1)xy\text{”} = \max(0, x, y, x + y - 1)$$

be a tropical polynomial. Then $F_t = 1 + x + y + t^{-1}xy$ is the corresponding family

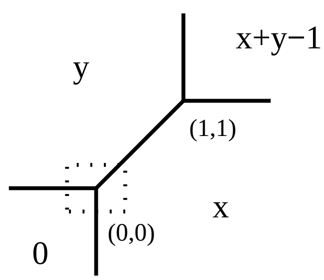


Figure 8. Near $(0, 0)$, this tropical curve looks like a line.

of classical hypersurfaces. When $t \to \infty$, $1 + x + y$ dominates, so the curve looks like a tropical line as in Figure 8 This happens when x, y are small as in $0 < r \leq |x|, |y| < R$.

Now consider the following change-of-coordinates,

$$\tilde{x} = \frac{x}{t}, \quad \tilde{y} = \frac{y}{t}.$$

The equation becomes

$$0 = 1 + t\tilde{x} + t\tilde{y} + t\tilde{x}\tilde{y}$$

and its zero-locus is the same as

$$0 = \frac{1}{t} + \tilde{x} + \tilde{y} + \tilde{x}\tilde{y}.$$

As $t \to \infty$, this becomes $0 = \tilde{x} + \tilde{y} + \tilde{x}\tilde{y}$. If we write $\overline{x} = \frac{1}{\tilde{x}}, \overline{y} = \frac{1}{\tilde{y}}$, we get the line

$$1 + \overline{x} + \overline{y} = 0.$$

This is the shape of the tropical variety near $(1,1)$. For t large, $\mathrm{Log}(V(F_t))$ looks like a thickening of the tropical variety.

7. non-Archimedean amoebas

The above patchworking argument can be formalized in algebra in terms of non-Archimedean amoebas, as originally done by Kapranov [11]. We begin with a field $\mathbb{K}$ with a valuation

$$\mathrm{val} : \mathbb{K} \to \mathbb{R} \cup \{-\infty\} = \mathbb{T}$$

satisfying

$$\begin{aligned} \mathrm{val}(ab) &= \mathrm{val}(a) + \mathrm{val}(b) \\ \mathrm{val}(a+b) &\leq \max(v(a), v(b)). \end{aligned}$$

We may define a non-Archimedean absolute value by $\|z\| = e^{\mathrm{val}(z)}$.

The field $\mathbb{K} = \mathbb{C}\{\{t^{\mathbb{R}}\}\}$ of generalized power series with real exponents is an example of such a field. This field consists of formal series of the form

$$z = \sum_{j \in J} a_j t^j$$

where $J \subset \mathbb{R}$ is a wel-ordered set and $a_j \in \mathbb{C}^*$. We define valuation by $\mathrm{val}(z) = -\min(J)$.

For $F = \sum_J a_J x^J$ a polynomial with $a_J \in \mathbb{K}$, the corresponding tropical polynomial is $f =$ "$\sum_J \mathrm{val}(a_J) x^J$".

Theorem 7.1 (Kapranov)**.** *If $V_F \subset (\mathbb{K}^*)^n$ is the zero-locus of a polynomial over $\mathbb{K}$ then the valuation $\mathrm{val}(V_F) \subset \mathbb{R}^n$ depends only on the valuation of the coefficients of F. In fact, $\mathrm{val}(V_F)$ is the tropical hypersurface defined by the tropical polynomial f.*

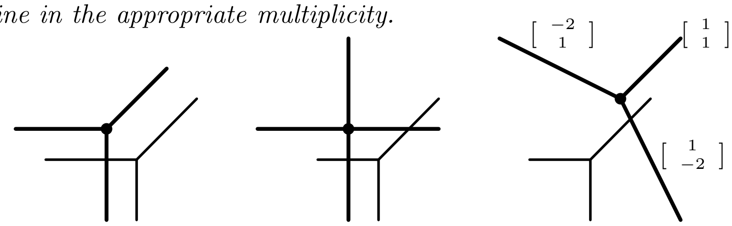
The non-Archimedean point of view allows one to talk about the phase for a tropical hypersurface. For $z = \sum_{j \in J} a_j t^j \in \mathbb{K}^*$, define the leading coefficient $\mathrm{lc}(z) = a_j$ where $j = \min(J)$. So we have in addition to a map $\mathrm{val} : \mathbb{K}^* \to \mathbb{T}^*$, a map $\mathrm{Arg} : \mathbb{K}^* \to S^1$ given by $\mathrm{Arg}(z) = \frac{\mathrm{lc}(z)}{|\mathrm{lc}(z)|}$. Therefore, one can look at the image of V_F under the map $(\mathrm{val}, \mathrm{Arg})$ to $(\mathbb{T}^*)^n \times (S^1)^n = (\mathbb{C}^*)^n$. Under certain genericity conditions on the coefficients, the whole image depends only on the valuations and arguments of the coefficients of F.

8. Local multiplicities

Any effective tropical cycle can be given a local multiplicity at any point.

Definition 8.1. *Let Z is a k-dimensional tropical cycle with all weights positive. For any integral basis of the ambient $\mathbb{Z}^n$, $e_1, \ldots, e_n$, consider the set of vectors $\{e_1, \ldots, e_n, -(e_1 + \cdots + e_n)\}$. By considering the union of the positive spans of each $(n-k)$-tuple of these vectors, we get an integral polyhedral complex. By giving each cone multiplicity 1, we get an $(n-k)$-cycle W. The* local multiplicity *is the minimum of the intersection number $W \cdot Z$ taken over all bases.*

Example 8.2. *Here are some examples of cycle of local multiplicities,* 1, 2, *and* 3 *respectively. The cycles are in bold. One can compute that they intersect the un-bolded line in the appropriate multiplicity.*



A cycle of local multiplicity 1 is locally irreducible.

9. Cycles of multiplicity 1

Let us consider fans that are cycles of multiplicity 1. By combining results of Ardila-Klivans [3] and of Mikhalkin and Sturmfels-Ziegler, these correspond to matroids.

Definition 9.1. *A matroid is a finite set E together with a rank function on the power set 2^E, $\mathrm{rk} : 2^E \to \mathbb{Z}_{\geq 0}$. The rank function satisfies the following properties:*

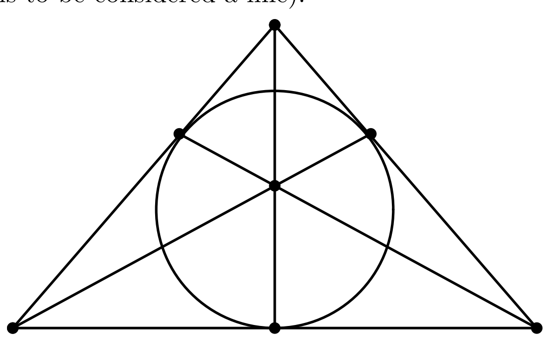
(1) *If $A \subset E$, then* $\mathrm{rk}(A) \leq |A|$,
(2) *If $A \subset B$, then* $\mathrm{rk}(A) \leq \mathrm{rk}(B)$,
(3) $\mathrm{rk}(A \cup B) + \mathrm{rk}(A \cap B) = \mathrm{rk}(A) + \mathrm{rk}(B)$.

A matroid abstracts the notion of linear dependence. The fundamental example of a matroid is if E is a set of vectors in a linear space, and

$$\mathrm{rk}(v_1, \ldots, v_k) = \dim(\mathrm{span}(v_1, \ldots, v_k)).$$

However, not all matroids arise in this fashion. Those that do not are called non-realizable.

Here is an example of a non-realizable matroid. Consider the Fano configuration (where the circle is to be considered a line).



Let E be the seven points of the configuration. Define the rank function as follows: $\mathrm{rk}(A) = |A|$ for $|A| = 1, 2$; $\mathrm{rk}(A) = 3$ for $|A| \geq 4$ or for A consisting of 3 non-collinear points; and $\mathrm{rk}(A) = 2$ for A consisting of 3 collinear points. This matroid is realizable as coming from a vector configuration only over fields of characteristic 2.

Definition 9.2. *A* flat *of the matroid is a subset of E whose rank increases when any element is added to it.*

One should think of a flat as the set of all vectors v_i that lie in a subspace $V \subset \mathbb{R}^n$.

Given a subspace $V \subset \mathbb{R}^n$, one may associated a matroid as follows. Let $e_1, \dots, e_n$ be a basis for the dual $(\mathbb{R}^n)^*$. The inclusion $i : V \hookrightarrow \mathbb{R}^n$ induces a surjection $i^* : (\mathbb{R}^n)^* \to V^*$. For a subset $A = \{i_1, \dots, i_k\} \subset E = \{1, \dots, n\}$, let $\mathrm{rk}(A) = \dim(\mathrm{span}(i^* e_{i_1}, \dots, i^* e_{i_k}))$. Equivalently, if we set F_A to be the coordinate subspace given by $x_{i_1} = \cdots = x_{i_k}$, $\mathrm{rk}(A)$ is the codimension of $V \cap F_A$ in V. One may produce the tropical variety associated to V with just the data of the matroid giving a cycle of multiplicity 1. For each flat A, form the vector $\sum_{i \in A} e_i$. A flag of flats is an inclusion $A_1 \subset A_2 \subset \cdots \subset A_k$ such that $\mathrm{rk}(A_1) < \mathrm{rk}(A_2) < \cdots < \mathrm{rk}(A_k))$. Each flag of flats gives a cone given by the positive span of the corresponding vectors. The union of such cones is the tropical variety of V.

We may also go from a cycle Z of multiplicity 1 to a matroid. Pick an integral basis for $\mathbb{R}^n$ coming from the case where multiplicity is minimized. Let $E = \{e_1, \dots, e_n\}$. Any subset of E corresponds to a face F_A of $\mathbb{TP}^n$. The rank is defined by $\mathrm{rk}(E) = \dim(Z) - \dim(\overline{Z} \cap F_A)$.

Sincere there are non-relatizable matroids, there are effective cycles not realizable as the limit of amoebas of classical varieties.

10. Tropical Manifolds

We would like to introduce a notion of a tropical manifold. The analytic notion of a complex manifold is that a local neighborhood is isomorphic to an open set in $\mathbb{C}^n$. In tropical geometry, we locally have a Euclidean topology, but we would like to see a Zariski topology. We will describe tropical manifolds as semi-ringed spaces, so it will suffice to give a local description.

Let $U \subset \mathbb{T}^n$ be an open set. Let $f : U \to \mathbb{T}$. be a regular function.

Example 10.1. *Let us consider the function $f : \mathbb{T} \to \mathbb{T}$ given by*

$$f(x) = \text{“}ax^2 + bx + c\text{”} = \max(2x + a, x + b, c).$$

The set-theoretical graph as pictured in Figure 9 is not a tropical variety. Instead, we should consider the tropical graph $\Gamma_f = V_{\text{“}y+f(x)\text{”}} \subset U \times \mathbb{T}$ as pictured in Figure 10. Note that we must put a “+” in the definition since “−” does not make sense. The tropical graph always contains the set-theoretical graph.

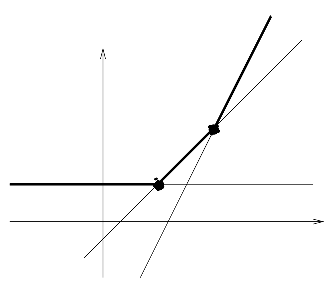


Figure 9. Set theoretic graph of f.

Define the principal open set $D_f = \overline{\Gamma_f} \cap (U \times \mathbb{R})$. We have a map $\Gamma_f \to U$ given by projection. One should think of this D_f has an open set in a sort of

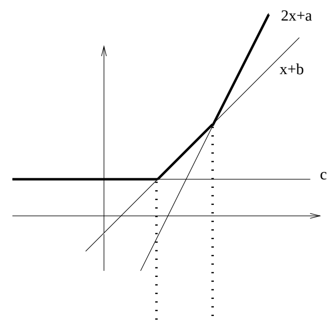


FIGURE 10. Tropical graph of f.

Grothendieck topology. The map $\Gamma_f \to U$ is said to be a tropical modification with center V_f.

Example 10.2. *Consider the function $f : \mathbb{T} \to \mathbb{T}$ given by $f(x) = x + a$. This function has tropical zero-locus set at $x = a$. Considering the analogous object over $\mathbb{C}$, we may take the complement of $z = a$ or alternatively, take the graph of $\frac{1}{f}$. In the tropical world, we may consider D_f. This is analogous to passing to the principal open set in the complex case.*

The definition of a tropical manifold is inductive on dimension. A zero-dimensional tropical manifold is a point with multiplicity 1.

Definition 10.3. *A n-dimensional smooth tropical manifold is a semi-ringed space $(X, \mathcal{O})$ that is of finite-type and such that any point $x \in X$ has a neighborhood U such that there is a finite sequence*

$$U = U_1 \xrightarrow{\tau_1} U_2 \xrightarrow{\tau_2} \cdots \longrightarrow \mathbb{T}^n$$

where each τ_i is a tropical modification with smooth center.

Remark 10.4. Each such neighborhood U is contained in some $\mathbb{T}^N$ as a tropical cycle. Moreover, these tropical cycles turn out to be as a complete intersections of the equations of the form "$y + f(x)$" and so are always approximated by complex amoebas by patchworking. In addition, the local multiplicity is always 1.

11. Smooth Tropical Curves

Let us examine tropical curves which are one-dimensional tropical manifolds. They will be graphs of a certain type. We first consider the case of local behavior modeled on $\mathbb{T}$. The integral affine structure on a tropical curve near a point modeled on $\mathbb{T}$ just consists of a vector field (up to sign change) along the curve. Moreover, the transition functions on this point of a curve are just translations of $\mathbb{T}$ which are isometries. Therefore, the integral affine structure near a point of the curve modeled on a point of $\mathbb{T}$ is equivalent to a metric on the underlying graph.

There are two types of points on $\mathbb{T}$, those in $\mathbb{T}^*$ and $-\infty$. A 1-valent vertex must map to $-\infty$. Any neighborhood of this point must have infinite length, so 1-valent vertices should be thought to be infinitely far away.

Now, let us consider modifications of tropical curves. A tropical modification at $-\infty$ does not change anything. In fact, the only functions on $\mathbb{T}$ whose tropical hypersurface is $-\infty$ are of the form "ax". The tropical graph of such a function is a line and hence is isomorphic to $\mathbb{T}$. A tropical modification at a point of $\mathbb{T}^*$ is given by $V_{\text{“}y+x+a\text{”}}$. This gives us a trivalent vertex as in Figure 11.

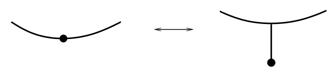


FIGURE 11. Tropical equivalence creates a trivalent vertex

Now let us consider a modification with the vertex of $\Gamma = V_{\text{“}y+x+a\text{”}}$. The zero-locus of a function f on Γ must be supported at the vertex with multiplicity 1 This function must satisfy

$$(df) \cdot v_x + (df) \cdot v_y + (df) \cdot v_{xy} = 1$$

where v_x, v_y, v_{xy} are primitive integer vectors along the three rays of Γ. Let $f =$ "$0 + x + y$". The restriction is 0 along two legs of the tropical line and linear with slope 1 on the third leg. After modification, we get a 4-valent vertex in $\mathbb{R}^3$ with rays along the directions $(-1,0,0), (0,-1,0), (0,0,-1), (1,1,1)$. We may continue to modify to get n-valent vertices in $\mathbb{R}^{n-1}$. In fact, all n-valent vertices that arise in this fashion are isomorphic.

It follows that a tropical curves can be viewed as metric graphs with leaves of infinite length and no 2-valent vertices. If Γ is a tropical curve, its genus is its first Betti number, $g = b_1(\Gamma)$. This turns out to be the dimension of the space of regular 1-forms on Γ.

12. Tropical point configurations

Proposition 12.1. *Any three distinct points on $\mathbb{TP}^1$ are equivalent to any other three distinct points (where equivalence is generated by a sequence of tropical modifications).*

Proof. Recall that $\mathbb{TP}^1$ is metrically isomorphic to $[-\infty, \infty]$. Once can find a sequence of tropical modifications that take any three given points to $-\infty, 0, \infty$. □

In fact, any four generic points (so that no triple lie on the same line) are tropically equivalent in $\mathbb{TP}^2$. In fact, we can use tropical modifications to put the four points at $[0 : -\infty : -\infty], [-\infty : 0 : -\infty], [-\infty : -\infty : 0], [0 : 0 : 0]$. In this configuration, we have already seen that the Fano plane axiom is satisfied. Thus it holds in general.

Four points in $\mathbb{TP}^1$ do have moduli. In fact, they have an invariant which is the cross ratio which is unchanged by tropical modification. Suppose the point p_1, p_2, p_3, p_4 are ordered on $\mathbb{TP}^1$. The cross-ration is defined as follows: connect p_1 to p_3 by a segment; connect p_2 to p_4. The cross-ratio is the length of the intersection as in Figure 12.

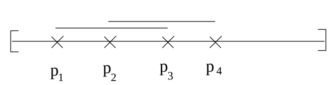
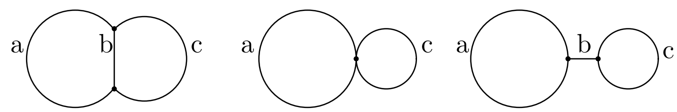


FIGURE 12. Computing a Tropical Cross-Ratio.

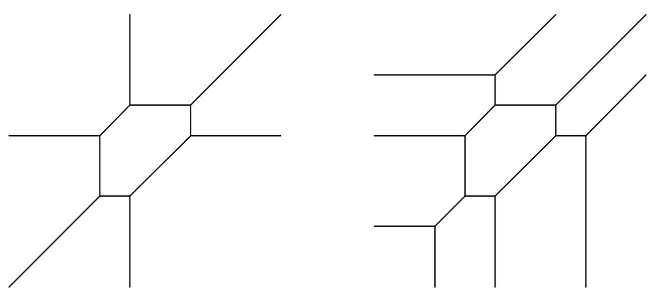
13. Moduli of Tropical Curves

Tropical curves are metric graphs without bivalent vertices such that each leaf has infinite length. The genus of a curve is its first Betti number. We only wish to study curves up to tropical modification. If $g > 0$, we may contract all leaves to find a compact model. If $g = 0$, the graph is a tree which is tropically equivalent to a point. If $g = 1$, the curve is topologically a circle and the only invariant is the length of the circle. If $g = 2$, there are three combinatorial types.



In two of the combinatorial types, there are three lengths a, b, c that can be varied. In the remaining, there are only two lengths. All genus 2 curves are hyperelliptic. They pictured curves have involutions give by reflection in a horizontal line.

In genus 3, not every curve is hyperelliptic, but they are all trigonal, that is, they have a degree 3 mapping to $\mathbb{TP}^1$. Not that these models can only be embedded into $\mathbb{R}^n$ after a tropical modification. Here are two different embeddings of the same genus 1 curve.

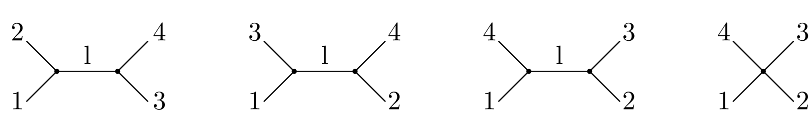


One can verify that each curve is cubic by computing its intersection number with a line.

Familiar facts about classical curves are also true for tropical curves. Any genus 1 curve can (after modification) be embedded into the plane as a tropical cubic while any non-hyperelliptic genus 3 curve can be presented as a quartic in $\mathbb{P}^3$.

We would like to define a moduli space $\mathcal{M}_{g,n}$ of genus g tropical curves with n marked points. Such a curve has n labeled infinite leaves. A 3-valent genus g curve with n leaves has $3g - 3 + n$ internal edges which must be assigned a length. Therefore, we expect $\mathcal{M}_{g,n}$ to have dimension $3g - 3 + n$. It has the structure of a polyhedral complex, moreover, it is a tropical manifold.

Let us work out the example of $\mathcal{M}_{0,4}$. It has curves of four combinatorial types



This shows that $\mathcal{M}_{0,4}$ consists of three rays corresponding to each of the three combinatorial types together with a vertex corresponding to the fourth. $\mathcal{M}_{0,4}$ is isomorphic to a tropical line. The following theorem is shown in [25]:

Theorem 13.1. *$\mathcal{M}_{0,n}$ is an effective cycle of local multiplicity* 1.

14. Embeddings of $\mathcal{M}_{0,n}$

Given a rational curve with n marked points, one has the cross ratio λ_{ijkl} for the four marked points i, j, k, l. By taking all unordered 4-uples of marked points, one obtains an embedding $\mathcal{M}_{0,n} \to \mathbb{R}^N$. This is studied by Gathmann, Kerber, and Markwig in [15].

One has another embedding by Plücker coordinates. Let us first consider a metric graph with finite-length leaves labeled by $\{1, \ldots, n\}$. The Plücker coordinate p_{ij} is the distance from end i to end j. This gives a point in $\mathbb{R}^{\binom{n}{2}}$. Now consider a point of $\mathcal{M}_{0,n}$. The leaves have infinite length. We may quotient the leaf length from $\mathbb{R}^{\binom{n}{2}}$. Let $\mathbb{R}^n$ (thought of as changing the lengths of the n leaves) act on $\mathbb{R}^{\binom{n}{2}}$ by the action

$$(l_1, \ldots, l_n) \cdot (p_{ij}) = (p_{ij} + l_i + l_j).$$

The image of $\mathbb{R}^n$is n-dimensional. So the embedding by Plücker coordnates gives a map

$$\mathcal{M}_{0,n} \to \mathbb{R}^{\binom{n}{2}}/\mathbb{R}^n.$$

15. Abel-Jacobi theory

Here we discuss tropical Abel-Jacobi theory. This is covered in detail in [26]. The genus of a tropical curve can also be defined as

$$g = \dim_R \Gamma(\Omega).$$

Here Ω is the sheaf of regular 1-forms on C. Given an embedding of C in $\mathbb{R}^n$, they are locally the restriction of a constant 1-form $\sum a_i \mathrm{d}x_i$ on $\mathbb{R}^n$.

There are two tropical Abelian varieties naturally associated to a tropical curve C. These are the Jacobian and the Picard variety, and they will prove to be isomorphic. The Picard group can be defined as the group of degree 0 divisors modulo linear equivalence. A divisor is a formal finite integer combination of points on C. The degree of a divisor is the sum of its coefficients. Each rational function f which is locally the difference of two convex functions with integral slopes defines a *principal divisor* (f) by taking its hypersurface. That is one locally looks at f as "$\frac{g}{h}$" and takes the formal difference of hypersurfaces. The principal divisors form a subgroup. We define linear equivalence by saying $D \sim D'$ if $D + (f) = D'$ for some rational f.

There is a combinatorial version of the Picard group on graphs coming out of the study of the chip-firing game. It has been studied in [9] and [8]. One has a graph togehter with a vertex designated as a bank. If one fires a vertex, it gives a chip to each of its neighbors. One tries to find a configuration in which every vertex but the bank is out of debt.

Classically the Jacobian of a smooth compact Rieman surface C is

$$J(C) = \Gamma(\Omega_C)^*/H_1(C, \mathbb{Z}).$$

Here $\Gamma(\Omega_C)^*$ is the dual to the space of regular 1-forms and $H_1(C, \mathbb{Z}) \subset \Gamma(\Omega)^*$ by taking $\gamma \in H_1(C, \mathbb{Z})$ to the functional

$$\omega \mapsto \int_\gamma \omega.$$

This definition carries over immediately to the tropical world. Here $\Gamma(\Omega)^* \equiv \mathbb{R}^g$ and is given a tropical structure by declaring a 1-form to be integral if it takes integer values on primitive integer tangent vectors. $H_1(C) \equiv \mathbb{Z}^g$ so $J(C) \equiv (S^1)^g$ as topological spaces. A precise tropical Torelli theorem is proven in [10]. It shows the extent to which a tropical curve is determined by its Jacobian.

Let $p_0 \in C$. We define the Abel-Jacobi map $\mu : \mathrm{Div}^d(C) \to J(C)$ as follows. If $D = \sum a_i p_i$, for each i, pick a path γ_i from p_0 to p_i. Define the functional $\mu(D)$ on $\Gamma(\Omega)$ by

$$\omega \mapsto \sum a_i \int_{\gamma_i} \omega.$$

The tropical Abel-Jacobi theorem states that the Picard group can be identified with the Jacobian variety.:

Theorem 15.1. [26] $\mathrm{Pic}^0(C) \equiv J(C)$.

By identifying the curve with $\mathrm{Div}^1(C)$, we may embed C in $J(C)$.

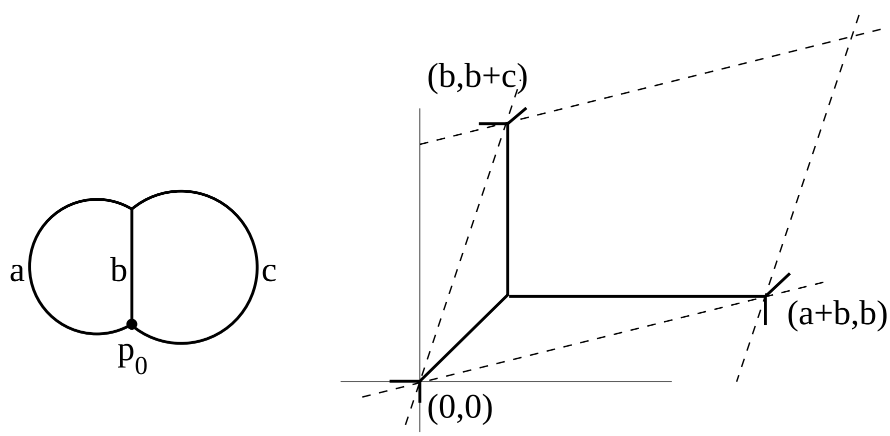


FIGURE 13. [26] $\mu(C)$ in the tropical Jacobian $J(C)$.

All Jacobians are examples of principally polarized abelian varieties. Being polarized is a symmetry condition which implies that the Jacobian is of the form $\mathbb{R}^g/\Lambda$ where the lattice Λ is given by a symmetric $g \times g$-matrix.

16. Riemann-Roch Theory

We want to study degree d maps $f : C \to \mathbb{TP}^1$. Such a map is determined up to translation by its poles and zeroes. A pair of divisors $D_0 = (p_1) + \cdots + (p_d), D_\infty = (q_1) + \cdots + (q_d) \in \mathrm{Sym}^d C$ are the zeroes and poles of a rational function if and only if $[D_0] - [D_\infty] = 0$ in $\mathrm{Pic}^0(C)$.

Consider the map $P : \mathrm{Sym}^d \times \mathrm{Sym}^d C \to \mathrm{Pic}^0$ given by $(D_0, D_\infty) \mapsto [D_0] - [D_\infty]$. This is a map from a $2d$-dimensional polyhedral complex to a g-dimensional complex. Assuming transversality assumption, we expect the preimage, $P^{-1}(0)$ to be $2d - g$-dimensional. Therefore, the space of maps $f : C \to \mathbb{TP}^1$ is $2d - g + 1$-dimensional. If we specify D_∞, we expect $P^{-1}(0) \cap (\mathrm{Sym}^d \times \{D_\infty\})$ to be $d - g$-dimensional. This implies that the expected dimension of the space of f with $(f) - D_\infty \geq 0$ to be $(1 - g) + d$-dimensional. This is the lower bound given by the Riemann-Roch theorem. Precise statements and proofs are given in [26] and [14]. They rely on a chip-firing argument introduced by Baker and Norine in [4].

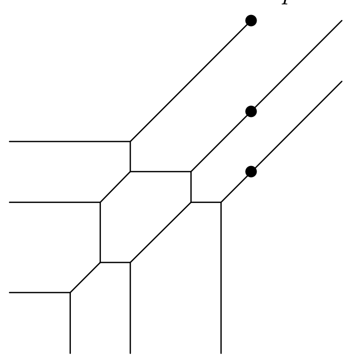
17. Maps of Curves to $\mathbb{TP}^n$

Suppose we have a genus g tropical curve C and a map $f : C \to \mathbb{TP}^1$. Let us compute the expected dimension of the space of deformations of the pair (C, f). The curve C belongs to the $(3g-3)$-dimensional moduli space $\mathcal{M}_g$. The map is expected to move in a $2d-g+1$-dimensional family. Therefore, we should expect the space of pairs (C, f) to be $(2d+2g-2)$-dimensional. A similar analysis can be applied to maps $f : C \to \mathbb{TP}^n$ by considering f to be an n-tuple of rational functions. The space of such maps is at least $d(n+1)+(1-g)(n-3)$. The dimension may be higher if conditions imposed by the map are not transverse.

Definition 17.1. *A curve $h : C \to \mathbb{TP}^n$ is called* regular *if the space of deformations of the pair (C, f) is exactly $d(n+1)+(1-g)(n-3)$. It is called* superabundant *otherwise.*

See [23] for more details.

Example 17.2. *Let $h : C \to \mathbb{TP}^2 \to \mathbb{TP}^3$ be given by composing the embedding of a plane cubic by a linear embedding L of $\mathbb{TP}^2$ in $\mathbb{TP}^3$. Suppose that the tropical hypersurface of L intersects the cubic in the three points marked points.*



Since the cycle in the cubic is planar, the condition that the cycle closes up imposes two conditions rather than three. The dimension of the deformation space is therefore higher than expected.

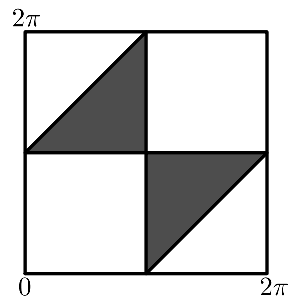
18. Phase tropical curves

We introduce phase tropical curves for 3-valent curves. The general theory [20] is only slightly more complicated. A tropical curve can be viewed as a pair of pants decomposition of a Riemann surface. Each vertex together with its adjacent edges form a tripod which may be viewed as a pair of pants. The tropical curves tells one how to glue the pairs of pants. We may also include data specifying how the legs and waist of the pants glue together. This is the twist parameter and is the imaginary part of the Fenchel-Nielsen coordinates.

We take as our model for our pair of pants the line H cut out by $z+w+1 = 0$ in $(\mathbb{C}^*)^2$. There are three punctures in this line corresponding to the missing coordinate axes in $\mathbb{P}^2$. This gives us three boundary circles. They will be denoted by $B_v(e)$ where e is the edge of the tripod corresponding to the boundary circle. The image of H under the argument map

$$\mathrm{Arg} : (\mathbb{C}^*)^2 \to (S^1)^2$$

is the alga below:



The three boundary circles of the alga correspond to the boundary circle of H. The boundary circles may be parameterized by

$$\phi \mapsto (\theta_j + \frac{p_j}{q_j}\phi).$$

The vector $\vec{s} = \mathrm{lcm}(q_1, \ldots, q_n)(\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}) \in \mathbb{Z}^n$ is called the boundary slope of s.

Definition 18.1. *A phase-tropical structure ρ on a tropical curve C is a choice of line $H(v) \subset (\mathbb{C}^*)^2$ for each vertex v together with an orientation-reversing isometry of boundary circles $\rho_e : B_v(e) \to B_{v'}(e)$ for each edge e connecting vertices v, v'.*

Definition 18.2. *A multiplicatively affine-linear morphism is a map*

$$A : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^m$$

given by

$$(z_1, \ldots, z_n) \mapsto (b_1 z_1^{a_{11}} \ldots z_n^{a_{1n}}, \ldots, b_m z_1^{a_{m1}} \ldots z_n^{a_{mn}})$$

for $b_i \in \mathbb{C}^*, a_{ij} \in \mathbb{Z}$.

A descends to a map on real tori

$$\alpha : (S^1)^n \to (S^1)^m$$

given by

$$(\zeta_1, \ldots, \zeta_m) \mapsto (\mathrm{Arg}(b_1) + a_{11}\zeta_1 + \cdots + a_{1n}\zeta_n, \ldots, \mathrm{Arg}(b_m) + a_{m1}\zeta_1 + \cdots + a_{mn}\zeta_n).$$

We may write $A|_{B_v(e)}$ for the restriction of α to the boundary circle corresponding to $B_v(e)$ in the alga.

Definition 18.3. *For a line $H(v) \subset (\mathbb{C}^*)^n$, and a boundary circle $B_e(v)$, $\sigma(e)$, the* phase of e *is the image of $B_v(e)$ under the composition of A with the natural quotient:*

$$(S^1)^2 \to (S^1)^n \to (S^1)^n / \langle \vec{s}(e) \rangle$$

where $\vec{s}(e)$ is the boundary slope. Note that $\sigma(e)$ is a point.

Definition 18.4. *A phase-tropical morphism $\Phi : (C, \rho) \to (\mathbb{C}^*)^n$ is a tropical morphism $C \to \mathbb{R}^n$ together with for every vertex v, an affine linear morphism $A_v : (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^n$ that restrists to $\Phi_v : H(v) \to (\mathbb{C}^*)^n$. For e adjacent to v, let $\Phi_{v,e} = \Phi_v|_{B_v(e)}$. In e connects vertices v, v', we require the condition that*

$$\mathrm{Arg} \circ \Phi_v = \mathrm{Arg}\, \Phi_{v'} \circ \rho_e : B_v(e) \to (S^1)^n.$$

One obtains a choice of phases from a phase tropical morphism. Let $\Phi_v : H(v) \to (\mathbb{C}^*)^n$. Let e_1, e_2, e_3 be the edges adjacent to v. The slope vectors $\vec{s}(e_1), \vec{s}(e_2), \vec{s}(e_3)$ must be coplanar in $\mathbb{Z}^n$. Suppose the slope vectors are non-collinear. Let S_v be the two dimensional subgroup of $(S^1)^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$ that contains the slope vectors. Once we choose an orientation, we may canonically identify $G_v = S_v/\vec{s}(e_j)$ with $S^1 = \mathbb{R}^1/(2\pi\mathbb{Z})$.

Definition 18.5. *We say the phase function σ are* compatible *at v if*

$$\sigma(e_1) + \sigma(e_2) + \sigma(e_3) = \mu(v)\pi$$

in G_v where $\mu(v) = |\vec{s}(e_1) \wedge \vec{s}(e_2)|$.

It turns out that a phase-tropical morphism defines a compatible phase-function σ on the flags of C.

We have the following approximation theorem:

Theorem 18.6. *Any phase-tropical morphism $h : C \to \mathbb{R}^n$ based on a regular tropical cure can be approximated by holomorphic maps $H_t : V_t \to (\mathbb{C}^*)^n$.*

Corollary 18.7. *Any regular tropical curve $h : C \to \mathbb{R}^n$ is a limit of holomorphic amoebas.*

We may use these results to represent curves in space as real algebraic curves.

Superabundant curves may not be approximable. An example can be derived from our superabundant cubic. Wiggle the marked points so that they do not lie on a tropical line. Then one has a genus 1 spatial cubic. Classical algebraic geometry says that such a curve must be planar.

This approximation theorem is an ingredient of a theorem lets one count classical curves through so many points by considering tropical curves through that many points. Using the technology of phase tropical curves, one can also count real curves and get Welschinger invariants.

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