Jan 20 '10

Higson II

Recall: continuous fields of Hilb. spaces.

Defn: X topo. space. H cont. field of Hilb spaces if

- $H = \{H_x\}_{x \in X}$
- $\Gamma(H)$ vector sp. of sections.
  - (a) $s \in \Gamma$ then $x \mapsto \|s(x)\|$ is conti fn on X
  - (b) $x \in X$ then $\{s(x) : s \in \Gamma(H)\} \subseteq H_x$ dense
  - (c) t (set theoretic section) with prop: $\forall x, \forall \epsilon$ there is nbhd U of x and $s \in \Gamma(H)$ st. $\sup_{y \in U} \|t(y) - s(y)\| \leq \epsilon$ then $t \in \Gamma(H)$

Rmk: If (a), (b) hold then $\exists!$ enlargement of $\Gamma(H)$ s.t. (c) holds.

Example: $M \longrightarrow X$ submersion.
Take $H_x = L^2(M_x)$ and $\Gamma(H)$ = smooth fns with cpt support on M.
(need smoothly varying family of Lebesgue measures)

Example: Constant fields.

Defn: A field is trivial if it is iso to a constant field

Rmk: We can restrict fields and pull back along maps.
We have locally trivial fields, but not all are such.

Counter-Example: Let $Y \subseteq X$ open where X cpt.
Given field H on Y we can push it forward to X by defining $H_x = 0$ for $x \in X \setminus Y$.
$\Gamma(\text{push-forward}) = \{s \in \Gamma(H) \mid \lim_{y \to \infty} \|s(y)\| = 0\}$.

Hermitian vector bundles are continuous fields over a cpt X. It is known that such vect. bundle are ~~[illegible]~~:

summands of trivial Hermitian bundle.

Stabilization (Kasparov) Every cts field is a summand of a trivial field.

Rmk: We can arrange: $H \oplus$ trivial = trivial. hence H's don't have interesting K-theory.

For example: $C_c^\infty(U)$ is a summand of $C^\infty(M) \oplus C^\infty(M) \oplus$ etc. (using partition of unity).

Kasparov's version of Atiyah-~~Singer~~ Jänich for X loc. cpt:

$$K^0(X) = \text{htpy classes of Fredholm operators on cts fields on } X.$$

Dirac-type Operators

$\mathbb{Z}_2$-grading

we'll work with $D = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$ formal adjoint

abuse notation

Grading operator: $\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

If M on which D defined is closed then D is essentially self-adjoints and has cpt resolvent.

This means we have orthonormal basis $\{u_n\}$

$$D u_n = \lambda_n u_n$$
$$|\lambda_n| \longrightarrow \infty$$

Note: $\|D \pm iI\| \geq 1$ hence $D \pm iI$ essentially 1:1.

It also has dense range. so $(D \pm iI)^{-1}$ exists.

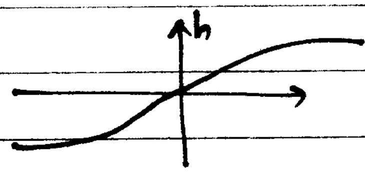
In fact, for h cts bounded on $\mathbb{R}$ we can define $h(D)$ bounded so that

$$h_1(D) h_2(D) = h_1 h_2(D)$$
$$h = (x \pm i)^{-1} \Rightarrow h(D) = (D \pm iI)^{-1}$$

__Recall__ Atiyah's observation: $[F,f]$ cpt

then $\begin{pmatrix}0 & F^*\\ F & 0\end{pmatrix} = h(D) + \text{cpt} + 1$

for any



recall that F is from polar decomp of D.

h "like" arctan.

__Example:__ $M \to X$ submersion and we have a family of Dirac-type ops. If M cpt then we have family of operators which is cpt in the sense of operators on families. (D_x on each M_x as in above discussion).

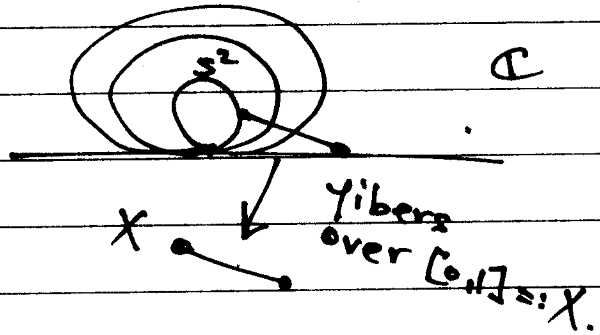
If M __not__ cpt there may be problems.

- D is essentially self adj if e.g. M complete
- If f is C_0 fn on M then $f h(D)$ is cpt for C_0 fn h on $\mathbb{R}$.

__Example:__ $D = \begin{pmatrix}0 & \bar\partial^*\\ \bar\partial & 0\end{pmatrix}$ on $M = \mathbb{C}$

But $h(D)$ not cpt $h \in C_0(\mathbb{R})$. So D not Fredholm and there is no index.

We could consider however $\begin{pmatrix}0 & \bar\partial + \bar\partial r^2\\ \bar\partial + \bar\partial r^2 & 0\end{pmatrix}$ r-radius



there is a Fredholm family interpolating between $\bar\partial$ on $M_0 = S^2$ and $\bar\partial + \bar\partial r^2$ on $\mathbb{C} = M_1$ hence the indices of $\bar\partial$ and $\bar\partial + \bar\partial r^2$ are equal.

<u>Example</u>: (Dirac Spinors)

V vect sp. (dim = 2k, 4k)

Cliff(V) Cliff alg.

S basic repn

$D = \sum e_j \partial/\partial x_j$ Dirac op on $C_c^\infty(V,S)$.

D is <u>not</u> Fredholm.

We could let D act on $C_c^\infty(V,S) \otimes S^* \cong C_c^\infty(V, \mathrm{End}(S))$

$$\cong C_c^\infty(V, \mathrm{Cliff}(V))$$

Let $c : V \longrightarrow \mathrm{Cliff}(V)$

$$c(v) = \varepsilon \cdot v$$

Let c act by right multi

Then $D + C$ <u>is</u> Fredholm and has index 1.

$$(D+C)^2 = \Delta + r^2 + (N - 2k)$$

N = number operator on Cliff(V)

<u>Example</u>: M^{2k} spin^c mfld

S spinor bundle.

$$V \xrightarrow{\pi} M$$

More generally, suppose V is a spin^c vect. bun./M with spinor bundle S. We can buid a cts field on V.

$$H_v = S_{\pi(v)} \quad v \in V$$

and a Fredholm op

$$D_v = \text{Clifford multi by } v \text{ (times } \varepsilon)$$

This has $h(D)$ opt, so there is an index in $K(V)$

This is $\quad \mathrm{Th}(S) \in K(V)$

Suppose M is spin^c mfld

Suppose $M \hookrightarrow V$ Euclidean sp. even dim.

Let $N_V M$ = normal bundle = spin^c str.

We get

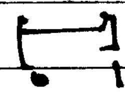
$$Th(S^{+}) \in K(N_V M)$$

This "is" the topological index of M.

(suppose to equal analytical index).

$N_V M$

$\downarrow$



Consider

$$\mathbb{N}_V M = N_V M \times \{0\} \amalg V \times [0,1]$$

it is a smooth mfld s.t.

$$\mathbb{N}_V M \longrightarrow V \times [0,1]$$

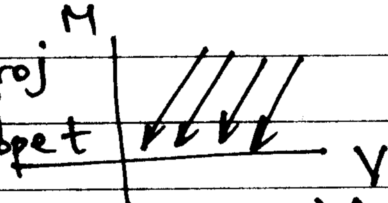
is smooth diffeo away from 0.

If $f : V \to \mathbb{R}$ smooth & $f|_M \equiv 0$ then

$$\tilde{f} : \mathbb{N}_V M \longrightarrow \mathbb{R}$$
$$(v,t) \longrightarrow \tfrac{1}{t} v$$
$$(X,0) = Xf$$

$\tilde{f}$ is smooth.

We have a submersion



$$M \times V \times [0,1] \qquad (m,v,t)$$
$$\downarrow \qquad\qquad\qquad \downarrow$$
$$\mathbb{N}_V M \qquad (m+tv,t)$$
or
$$(P_m(v),0)$$

Now consider (Fiberwise Dirac) $\hat{\otimes}\, 1 + 1 \otimes$ Cliff multi by C

acting on sections of $S(M) \otimes S_V$ spinor vector sp for V.

We obtain an index $\in K(\mathbb{N}_V M)$

$$N_V M \hookrightarrow \mathbb{N}_V M \quad (t=0)$$
$$V \hookrightarrow \mathbb{N}_V M \quad (t\neq 0)$$

@ $t=0$, $\text{Index} = Th(S^*) \in K(N_v M)$

@ $t=1$, $\text{Index} = \text{Index}(D) \cdot Th(S) \in K(V)$

When dim $4k$ you get

$$\text{Index}(D) = \text{topo. index}.$$