1 Preface

Two problems involving flat connections over manifolds:

- computing monodromy of Schrödinger operators on surfaces,
- computing Chern-Simons invariants of flat connections.

Unifying tool useful for both: *abelianization*. Arose in SUSY gauge theory (work with Gaiotto, Moore) but suppress that in this talk.

2 Setup

$M$ a manifold: a $G$-connection $\nabla$ over $M$ is a principal $G$-bundle $P$ plus a notion of parallel transport: $F_{\nabla}(\varphi) \in \text{Hom}(P_{x}, P_{x'})$.

Call $\nabla$ **flat** if $F_{\nabla}(\varphi)$ depends only on homotopy class of $\varphi$. Two connections equivalent if have map of principal bundles $P \to P'$ commuting with the $F$.

If we fix a section $s$ of $P$ ("fix a gauge") then $F_{\nabla}(\varphi)$ takes $s(x)$ to $gs(x')$ for some $g \in G$; or in short, $F_{\nabla}(\varphi) \in G$. In particular, fixing a basepoint $x$ this gives a rep $\pi_1(M) \to G$. There is an identification of moduli spaces

$$\mathcal{M}(M, G) = \{\text{flat $G$-connections}\}/\sim = \{\text{reps $\pi_1(M) \to G$}\}/\sim$$
3 Abelianization on $S^1$

Take $G = GL_N\mathbb{C}$.

First $M = S^1$. Then a flat $G$-connection $\nabla$ over $M$ means a matrix,

$$A \in GL_N\mathbb{C}$$

If $A$ is generic, it can be diagonalized, in finitely many ways. Eigenvalues give local coordinates on $\mathcal{M}(M, G) = GL_N\mathbb{C}/\sim$.

In terms of $\nabla$ this means: we can trivialize $P$ in such a way that all $F_{\nabla}(\varphi)$ are diagonal.

4 Abelianization on punctured torus

How about two matrices?

$$A, B \in GL_N\mathbb{C}$$

modulo simultaneous conjugation. Let’s think of $A, B$ as giving a representation

$$\pi_1(M) \rightarrow GL_N\mathbb{C}$$

where $M$ is the once-punctured torus. i.e. a flat connection $\nabla$ over $M$.

If $A, B$ don’t commute, hopeless to simultaneously diagonalize them. So, we won’t be able to find a global diagonal gauge. But there’s something you can do.

Take $N = 2$ for a minute, so $G = GL_2\mathbb{C}$. Now draw a picture $\mathcal{W}$ on $M$:

We can find a gauge off $\mathcal{W}$ such that:

- parallel transports off $\mathcal{W}$ are diagonal;
parallel transports across dashed lines are strictly off-diagonal;
parallel transports across solid lines are upper-triangular unipotent.

Call such a gauge an *abelianization* (wrt $\mathcal{W}$). Two abelianizations are equivalent if they differ by a diagonal gauge transformation.

Facts:

- a generic $A, B$ can be abelianized in 2 ways, up to equivalence.
- The diagonal/off-diagonal parallel transports assemble into an almost-flat $GL_1(\mathbb{C})$-connection $\nabla^{ab}$ over a branched double cover $\Sigma \to M$.

Thus we get an element $\mathcal{X}_\gamma \in \mathbb{C}^\times$ (up to sign) for each $\gamma \in H_1(\Sigma, \mathbb{Z})$. These are analogues of the *eigenvalues* for a single matrix. They give local coordinates on moduli space $\mathcal{M}(C, G)$ of pairs $A, B$ (identify it with $(\mathbb{C}^\times)^5$).

The abelianization amounts to an isomorphism

$$\iota : \nabla \simeq \pi_* \nabla^{ab}$$

off the walls, which has unipotent jumps.

Similar statements for any punctured surface with an ideal triangulation, and for any $N$. eg for $N = 2, 3$:

Properties:
the $\mathcal{X}_\gamma$ give local Darboux coordinates for Atiyah-Bott-Goldman Poisson structure on moduli spaces of representations: $\{\mathcal{X}_\gamma, \mathcal{X}_\mu\} = \langle \gamma, \mu \rangle \mathcal{X}_{\gamma+\mu}$. for $N = 2$, complexified shear coordinates. for higher $N$, related to cluster coordinates [Fock-Goncharov]

changing triangulations gives coordinate systems related by cluster transformation, like $(x, y) \rightarrow (x(1 + y), y)$.

invariants of the representation like $\text{Tr } ABABA^{-1}$ are finite sums of $\mathcal{X}_\gamma$. (So for each loop on $M$ there’s corresponding finite bunch of loops on $\Sigma$.)

5 Opers

Now fix a Riemann surface $M = C$ and a meromorphic quadratic differential $\phi_2$ on $C$. This determines Schrödinger operators: diff ops $D_{\hbar, \phi_2} : K_C^{-1/2} \rightarrow K_C^{3/2}$, locally like $\hbar^2 \partial_z^2 + \phi_2(z)$; analytic continuation of solutions gives flat $SL_2\mathbb{C}$-connections

$$\nabla_{\hbar, \phi_2}$$

and corresponding monodromy representation

$$\rho_{\hbar, \phi_2} : \pi_1(C) \rightarrow SL_2\mathbb{C}$$

A basic question: compute $\rho_{\hbar, \phi_2}$. Concretely, e.g. this could mean computing traces $\text{Tr } F_{\nabla_{\hbar, \phi_2}}(\phi)$. 

For this: define *critical graph* $\mathcal{W}(\vartheta, \phi_2)$ as follows. $C$ has foliation with the leaves defined by condition: $e^{-i\vartheta}\sqrt{\phi_2}$ real. Singular at zeroes and poles of $\phi_2$. Now, suppose $\phi_2$ has only simple zeroes. Then 3-pronged singularity:

For $\vartheta$ generic, no critical leaf ends up on another zero. Then $\mathcal{W}(\vartheta, \phi_2)$ is union of critical leaves. For convenience, suppose $\phi_2$ has at least one pole of order $\geq 2$.

Then, facts (theorem in progress with Kohei Iwaki, related to work with Marco Gualtieri and Nikita Nikolaev):

- For $\hbar$ generic, $\nabla_{\hbar, \phi_2}$ admits a *canonical* abelianization, (with respect to the critical graph $\mathcal{W}(\vartheta = \arg \hbar, \phi_2)$), for which the double cover $\Sigma$ is spectral curve

$$\Sigma = \{\lambda^2 + \phi_2 = 0\} \subset T^*C$$

- This abelianization has

$$\mathcal{X}_\gamma(\hbar) \sim \exp(Z_\gamma/\hbar + \cdots)$$

as $\hbar \to 0$ (in $\frac{1}{2}$-plane centered around $\vartheta$), where

- $Z_\gamma = \oint_\gamma \lambda$, $\lambda$ Liouville 1-form on $T^*C$,
- $\cdots$ is a computable series in powers of $\hbar$.

Proof: a rigorous version of “exact WKB” method [Voros, Ecalle, …] using recent results on Borel summability [Koike, Schafke]. Walls are “Stokes curves.”
This result determines the asymptotic expansion of the traces. (They have Stokes phenomena, i.e. it depends on how $\hbar \to 0$: dominated by the term with largest $\Re(Z_{\gamma}/\hbar)$.) With more work, can try to bootstrap into an actual computation of the traces themselves.

Conjecture: similar picture for higher-degree equations. In this case each wall is carrying a label $ij$ which tells what kind of unipotent matrix goes there. New phenomenon: scattering of the walls.

In general the networks which appear here are not related to triangulations. Some interesting new combinatorics.

6 3-manifolds

Abelianization seems to be useful for 3-manifolds too (work in progress with Dan Freed).

The picture is similar to before: 3-manifold $M$, branched double covering $\Sigma \to M$, network $\mathcal{W} \subset M$ of codimension-1 walls, relation $\iota : \nabla \simeq \pi_* \nabla^{ab}$ off $\mathcal{W}$. But, a new phenomenon: the connections $\nabla^{ab}$ which we consider have to be singular at some circles $S_i \subset \Sigma$. (Double-covering pieces of wall connecting branch loci.)

Holonomy obeys relation:
7 Triangulated 3-manifolds

Basic example: an ideal triangulation of a 3-manifold (glue together tetrahedra along their faces, with vertices deleted.) For every such manifold there is an associated $W$. Double cover $X \to M$ now branched over a 1-manifold which threads through the tetrahedron. In the interior of the tetrahedron, two walls meet head-on, in a line segment, double-covered by a circle $S_i$.

\[ \text{e.g. } M = \text{figure-eight knot complement } S^3 \setminus K, \text{ glued together from 2 tetrahedra.} \]

Now want to study flat $SL_2\mathbb{C}$-connections over $M$, by abelianization. For example, $M$ is hyperbolic [Riley-Thurston], so it has a
particular flat $PSL_2\mathbb{C}$-connection $\nabla$. Can lift it to $SL_2\mathbb{C}$. Can we construct it? Not so trivial, since

$$\pi_1(M) = \langle A, B | A^{-1}BAB^{-1}AB = BA^{-1}BA \rangle$$

How to make matrices obeying these conditions? Idea: build it by starting with $\nabla^{ab}$. This means giving a class $\in H^1(\Sigma \setminus S, \mathbb{C})$ obeying the constraints at each $S_i$. Determined by its values $\mathcal{X}_i$ on loops around $S_i$.

But the $\mathcal{X}_i$ have to obey our constraints. This leads to a system of algebraic equations (Thurston gluing equations). e.g. for figure-eight knot complement, if we also ask for $PSL_2\mathbb{C}$-unipotent holonomy around the boundary torus:

$$\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{X}, \quad \mathcal{X}(1 - \mathcal{X}) = 1$$

thus

$$\mathcal{X} = e^{\pi i/3}$$

(NB, this is different from the surface case where we got a moduli space: here expected dimension is zero.)
8 Chern-Simons

Let $G = SL_N \mathbb{C}$. Suppose given a closed 3-manifold $M$ carrying a $G$-connection $\nabla$.

Then we can consider the (level 1) Chern-Simons invariant. Since $G$ has $\pi_0 = \pi_1 = \pi_2 = 0$ every $G$-bundle over $M$ is trivial, thus we can represent $\nabla$ as $d + A$ for $A \in \Omega^1(g)$, and then the Chern-Simons invariant is

$$CS(\nabla) = \frac{1}{8\pi^2} \int_M \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

The choice of trivialization makes a difference, but changes $CS(\nabla)$ by something in $\mathbb{Z}$, so what’s well defined is

$$CS(\nabla) \in \mathbb{C}/\mathbb{Z}$$

The figure-eight knot just described was slightly different: $M$ has boundary, $PSL_2 \mathbb{C}$-unipotent holonomy on boundary. Here, can similarly define $CS(\nabla) \in \mathbb{C}/\frac{1}{4}\mathbb{Z}$.

For the $\nabla$ coming from a hyperbolic structure,

$$CS(\nabla) = (\text{real}) + i \text{vol}(M)/4\pi^2$$

i.e. this is a “complexified volume.”

Question: how to actually compute $CS(\nabla)$?

9 Chern-Simons by abelianization

Theorem (in progress with Dan Freed): if we have $\mathcal{W}$ on $M$ and a flat $\nabla$ over $M$, abelianized by some $\nabla^{\text{ab}}$ over $\Sigma$, can compute $CS(\nabla)$. This computation is “easy” and gives $CS(\nabla) = CS(\nabla^{\text{ab}})$, except
for additional contributions from the $S_i$. The additional contribution is, *loosely speaking* $\frac{1}{4\pi^2}\text{Li}_2(\mathcal{X}_i)$.

Upon choosing a trivialization of the line bundle $L$ over $X$ this recovers dilogarithmic formulas for hyperbolic volumes. Choices of branch of $\text{Li}_2$ get dictated by trivialization of $L$.

For example, figure-eight knot complement: get

$$
CS(\nabla) = \frac{1}{4\pi^2}(\text{Li}_2(e^{\pi i/3}) + \text{Li}_2(e^{\pi i/3})) \approx 2.02988i
$$

recovering the hyperbolic volume (NB we take different branches for the two dilogs!)

Related to [Dupont, Neumann, Zickert]; higher rank version related to [Garoufalidis-Thurston-Zickert] in case of triangulations.

10 Chern-Simons and topological strings

What we said: $SL(N, \mathbb{C})$ Chern-Simons theory on $M$ related to $GL(1, \mathbb{C})$ Chern-Simons on $\Sigma \to M$, *deformed* by some funny singular behavior.

This deformation would arise naturally in a model topological string: if $\Sigma \subset T^*M$, and equip $T^*M$ with almost complex structure, get a deformation of Chern-Simons, where the $S_i$ are the boundaries of holomorphic discs in $T^*M$ [Witten, Ooguri-Vafa].

So hopefully all the story of abelianization has a natural meaning in topological string (Floer theory).

Key open question here: can we say something about *quantum* complex Chern-Simons using this idea?