1 Aim

I want to describe some recent progress in 4-dimensional quantum field theory.

Our universe seems to be well described by 4-dimensional quantum field theory. By now (thanks to enormous effort in the 20th century) we think we know the rules of the QFT game pretty well, and we know a lot about which QFT describes our particular universe. But even easy-sounding questions are hard to answer — e.g. to go from the fundamental equations of the theory to “observables” like the spectrum of particles one would actually detect.

Theorists often study “toy models” — examples of QFT’s which are not directly realized in our universe — as a way to get general lessons about what can happen in QFT.

Recently our class of toy models has been dramatically expanded, thanks to a surprising trick. The trick, in short, is that in order to study four-dimensional physics, it is convenient to start from a six-dimensional perspective, then take two of the dimensions to be small. [Witten, Gaiotto, Gaiotto-Moore-Neitzke] This is the origin of the “4 + 2 = 6” in my title. From this perspective, many “mysterious” phenomena in 4d physics become easier to understand, or at least easier to predict.
2 Maxwell theory

Let’s start with vacuum Maxwell’s equations (in units where \( c = 1 \)):

\[
\begin{align*}
\text{div} \, \vec{E} &= 0, \\
\text{curl} \, \vec{B} - \frac{\partial \vec{E}}{\partial t} &= 0, \\
\text{div} \, \vec{B} &= 0, \\
\text{curl} \, \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0.
\end{align*}
\]

These equations are linear in the fields. So, we can actually solve them exactly: basis for solutions given by propagating electromagnetic waves (light). Linearity means that we can superpose these waves: no interactions. Linearity also means that the quantum version of this theory is understandable. For example, the ”Hilbert space of 1-particle states” in this theory just consists of states \( | \vec{k}, \vec{\epsilon} \rangle \) representing quantized electromagnetic waves (photons) with momentum \( \vec{k} \) and polarization \( \vec{\epsilon} \).

These equations also have a remarkable symmetry to them: they would be the same if we exchange

\[
\vec{E} \to \vec{B}, \quad \vec{B} \to -\vec{E}
\]

This is the fact of electric-magnetic duality.

What is the use of this duality? Suppose we couple the theory to external charges. The duality says for example that if we have two monopoles separated by some distance, with charge \( p_1, p_2 \), we can compute the potential energy between them: by duality, it must be the same as the potential energy between two electric charges,
\[ V = -p_1 p_2 / r! \] And similarly for any other question we ask about complicated combinations of electric and magnetic charges.

This duality also survives into the *quantum* version of Maxwell theory – not too surprising. But, there’s a little twist. Recall Dirac’s quantization law: fix an electrically charged particle of charge \( q \), and a magnetic monopole of charge \( p \); then \( pq \) must be an integer (in units where \( \hbar = 1 \)). This means that if \( q_0 \) and \( p_0 \) are the minimum quanta of electric and magnetic charge then \( p_0 = 1 / q_0 \). Thus, if the force between elementary electric charges is small (weak coupling), the force between elementary magnetic charges will be large (strong coupling). So, electric-magnetic duality is in some sense a strong/weak duality.

### 3 Gauge theory

The Standard Model of particle physics is founded on equations which are in some ways analogous to Maxwell’s equations. The electromagnetic field-strengths \( \vec{E}, \vec{B} \) are replaced by vectors of *matrices*. For the weak interactions we should take \( 2 \times 2 \) skew-Hermitian matrices, for strong interactions \( 3 \times 3 \) ones. Then we can write the equations of *classical Yang-Mills theory*:

\[
\text{div} \, \vec{E} = g[\vec{A} \cdot \vec{E}], \quad (3.1)
\]

\[
\text{curl} \, \vec{B} - \frac{\partial \vec{E}}{\partial t} = g[V \vec{E}] - g[\vec{A} \times \vec{B}], \quad (3.2)
\]

\[
\text{div} \, \vec{B} = -\frac{1}{2} g \text{div}[\vec{A} \times \vec{A}], \quad (3.3)
\]

\[
\text{curl} \, \vec{E} + \frac{\partial \vec{B}}{\partial t} = -\frac{1}{2} g \partial_t [\vec{A} \times \vec{A}] + g \text{curl}[V \vec{A}]. \quad (3.4)
\]
These equations involve both the fields $\vec{E}$, $\vec{B}$ and their potentials $(V, \vec{A})$. $\vec{E}$ and $\vec{B}$ are determined by $(V, \vec{A})$ by equations similar to the usual ones, with corrections. But, no way to write the equations just using $\vec{E}$, $\vec{B}$. The notation $[\vec{A} \cdot \vec{E}]$ means $\sum_{i=1}^{3} A_i E_i - E_i A_i$, and similarly for the other [] terms.

The equations are *nonlinear*: much more difficult to solve! Reflects the fact that these “gauge fields” actually *interact* with one another. The parameter $g$ controls the strength of this interaction.

One could hope to solve the equations by some kind of perturbation expansion in $g$.

Similarly for the quantum theory: perturbation in $g$ is possible in principle (not easy), but as we increase $g$ things will get harder and harder.

In particular, questions like “what are the 1-particle states in the quantum version of this theory?” are very difficult indeed! It’s generally believed that there are no massless particles in this theory: this is the phenomenon of *confinement*. But a really satisfactory theoretical understanding of confinement is lacking (1 million dollars for mathematical proof). In some sense the trouble is that one doesn’t adequately understand the physics at large $g$.

### 4 Electric-magnetic duality, redux

OK, so how can we understand the physics at large $g$?

Electric-magnetic duality symmetry is far from obvious in the Yang-Mills equations. It was proposed [Montonen-Olive] that there is actually a hidden symmetry in the *quantum* theory, which again exchanges $\vec{E} \leftrightarrow \vec{B}$, and *simultaneously* exchanges $g \leftrightarrow 1/g$ (strong/weak
duality again!)

A bizarre statement: says that the theory “becomes weakly coupled again,” when we go to very large $g$.

It’s so bizarre that nobody believed it. And for good reason: it is not true.

But, the idea was too good to be completely wrong. It turns out that it is true in a supersymmetric extension of Yang-Mills theory. This is a theory in which we have not only the gauge fields but also some additional matter built in, in a very specific way. The version we want is $N = 4$ supersymmetric Yang-Mills theory, where the $N$ is measuring how much extra symmetry the theory has, and also roughly measuring how much extra matter we had to add — multiplied the physical polarizations by 8. Think of this as a kind of ”toy model” for the non-supersymmetric Yang-Mills theory we really want to undersand.

This ”non-abelian” version of electric-magnetic duality is called $S$-duality. Lots of evidence by now that $S$-duality is actually true [Sen, Vafa-Witten]; but still somewhat mysterious.

5  A new picture

Now, a new way of making the electric-magnetic duality look ”easy”.

It requires us to swallow one claim which comes from string theory (but much less than the full string machine), as follows.

• There exists an interacting, scale invariant six-dimensional supersymmetric quantum field theory. (”Theory X”).
Formulate Theory X on a space-time which is $S^1 \times \mathbb{R}^5$, where $S^1$ has radius $R$, and then look at the theory at low energies ($E \ll 1/R$). We get a five-dimensional version of supersymmetric Yang-Mills theory, with coupling strength $g_5 = \sqrt{R}$.

Then, standard QFT method: further compactification on another $S^1$ of radius $R'$ gives $N = 4$ supersymmetric Yang-Mills theory with coupling strength $g = g_5/\sqrt{R'} = \sqrt{R/R'}$.

But this is strange — the same theory can also be viewed as $N = 4$ super Yang-Mills with coupling $g = \sqrt{R'/R}$! (Draw the diagram.)

Conclusion: the ”easy” geometrical symmetry exchanging the two circles in $T^2$ corresponds to the ”deep” $S$-duality. [Vafa]

6 Closer to reality

How about less supersymmetric theories?

In the last few years it’s been understood that a very similar picture can be applied to theories with “$N = 2$ supersymmetry,” too. (Still too much for the real world, but getting closer. Only multiplied the number of physical polarizations by 4.) [Gaiotto]

Just replace the 2-torus by other surfaces $C$! i.e. take Theory X in a spacetime of the form $C \times \mathbb{R}^4$. In this way we can realize all kinds of different 4-dimensional quantum field theories just by varying our choice of $C$. For example: suppose we take $C$ to be a sphere with 4 punctures, (and let our matrices be $2 \times 2$). Then we obtain $N = 2$ supersymmetric Yang-Mills theory (coupled to some matter). The coupling of the theory is determined by the position of the 4 punctures.
In fact, just as for the torus, we can obtain \( N = 2 \) super Yang-Mills in many different ways. Each choice of a curve which separates the 4 punctures into 2 pairs gives an example, with a different value of the coupling. Thus we get a large number of ”S-dualities” relating different versions of the theory, with different couplings.

More generally, get lots of QFT’s which hadn’t been known before: stuff much wilder than Yang-Mills, only beginning to be explored. And a huge number of S-dualities between them. [Gaiotto, Distler-Chacaltana] And at the same time we get a lot of new tools for exploring them...

7 The spectrum of particles

This stuff can be used to address some down-to-earth questions. Given any complex physical system, one of the basic questions you always want to ask is: what does it do? e.g. what is the ground state and what are the excitations around the ground state?

In the supersymmetric theories we have been discussing, the question what are the ground states was basically answered in 1995 [Seiberg-Witten]. The question what are the excitations is harder.

The geometric picture I have been describing leads to a sharp answer to this question, at least for the stable excitations, i.e. particles with the smallest possible mass consistent with their charge. Namely: such particles correspond to certain networks of strings on the surface \( C \) [Klemm-Lerche-Mayr-Vafa-Warner, Gaiotto-Moore-Neitzke]. This is something you can really study on the computer. (Show movies?)

This already leads to some surprises. For example: in joint work
with Tom Mainiero (student in Austin physics dept) and several collaborators at Rutgers [Galakhov-Longhi-Moore], we found that in the $N = 2$ supersymmetric version of the Yang-Mills theory with $3 \times 3$ matrices, there are vacua where the number of particles with mass $\leq M$ grows exponentially with $M$.

It’s surprising to us that this can happen at all in field theory! Thermodynamic consequences not clear at the moment...