1 Preface

Counts of “special trajectories” of quadratic differentials (saddle points and closed trajectories) are a well-studied subject. Recently it has become clear that they are also examples of “generalized Donaldson-Thomas invariants.”

That’s interesting in itself: a nice computable example in that theory. But embedding them into this context has also led to several “external” developments:

• They obey a wall-crossing formula written down by Kontsevich and Soibelman, which governs how the special trajectories appear and disappear as the quadratic differential is varied;

• They are important ingredients in a systematic scheme for analyzing the asymptotics of differential equations (WKB);

• They are also key ingredients in a new construction of hyperkähler (Ricci-flat) metrics;

• Maybe most interesting, they admit a natural generalization to “higher-rank” invariants — attached to any Lie algebra of type ADE (quadratic differentials are the $A_1$ case);

• They are part of the physics of $\mathcal{N} = 2$ supersymmetric quantum field theory.

In these talks I’ll try to describe all of this stuff from a sort of unified perspective. This perspective is work in progress with Davide Gaiotto and Greg Moore — an improvement of the approach we have described before. Some details can therefore be wrong but the basic picture is by now quite clear.
2 S-walls

Fix a compact complex curve $C$. We are going to do a construction involving quadratic differentials on $C$:

$$\varphi_2(z) = f(z) \, dz^2.$$

Any $\varphi_2$ determines a 1-parameter family of (singular) foliations $F(\varphi_2, \vartheta)$ of $C$. Leaves of $F(\varphi_2, \vartheta)$, or “trajectories”, are paths along which $e^{-i\vartheta} \sqrt{\varphi_2}$ is a real 1-form. (In local coordinates: write $\varphi_2 = dw^2$, then the leaves are straight lines of inclination $\vartheta$ in the $w$-coordinate.)

$F(\varphi_2, \vartheta)$ has singularities at the zeroes of $\varphi_2$. At simple zeroes, the singularity is 3-pronged. (Picture.) Assume for now that $\varphi_2$ has only simple zeroes. In essentially everything that follows, we will focus on the trajectories emerging from the zeroes. Call them “separating trajectories” or “$S$-walls.”

It may happen that an $S$-wall has both ends on a zero. In that case we call it a “special trajectory.” These can come in two flavors: either saddle connections or closed trajectories. (Picture.)

Our interest is in the question: how many special trajectories occur in $F(\vartheta, \varphi_2)$?

First observation: special trajectories can occur at most at countably many $\vartheta$.

Why? $\varphi_2$ determines a double cover of $C$,

$$\Sigma(\varphi_2) = \{ \lambda^2 - \varphi_2 = 0 \} \subset T^*C.$$ 

Each special trajectory of $\varphi_2$ can be lifted in a canonical way to a 1-cycle on $\Sigma(\varphi_2)$; let $\gamma \in \Gamma = H_1(\Sigma, \mathbb{Z})$ denote its homology class. Call $\gamma$ the “charge” of the trajectory.
Now, for any $\gamma \in \Gamma$ we can define
\[
Z_\gamma = \oint_\gamma \lambda
\]
with $\lambda$ the tautological 1-form on $T^*C$. If $\gamma$ is the lift of a special trajectory, then we must have $Z_\gamma \in e^{i\vartheta}\mathbb{R}_-$. But there are only countably many $\gamma \in \Gamma$, so this equation can be satisfied only for countably many $\vartheta$. Moreover, once we fix $\gamma$, $\vartheta$ is determined.

3 Punctures

To reduce potential analytic hazards, fix $n > 0$ marked points $z_1, \ldots, z_n$ on $C$ (“punctures”). Let $\mathcal{B}$ be the space of meromorphic quadratic differentials $\varphi_2$ on $C$, with double poles at all of the $z_i$. (I believe all of my main statements will be true even without these punctures, but at some moments I will rely on them to simplify the arguments; also, the simplest explicit examples are cases with punctures.) Let $\mathcal{B}' \subset \mathcal{B}$ consist of $\varphi_2$ with only simple zeroes.

In case with punctures, $\Sigma$ also has punctures: it is a double cover of $C \setminus \{z_1, \ldots, z_n\}$.

4 DT invariants

Now assume we are in the “generic” situation: all $Z_\gamma$ are linearly independent over $\mathbb{R}$. (This is a condition on $\varphi_2$.) In that case, the possible phenomena are relatively limited. Either isolated saddle connections, or pairs of closed trajectories, bounding an annulus.
We define

\[ \Omega(\gamma; \varphi_2) = \begin{cases} 
1 & \text{if } F(\varphi_2, \vartheta = \text{arg } -Z_{\gamma}) \text{ contains a saddle connection}, \\
-2 & \text{if } F(\varphi_2, \vartheta = \text{arg } -Z_{\gamma}) \text{ contains a closed trajectory}, \\
0 & \text{otherwise}. 
\end{cases} \]

So the \( \Omega(\gamma; \varphi_2) \) are “counting” the special trajectories, while keeping track of their topological types.

5 Wall-crossing

As we vary the quadratic differential \( \varphi_2 \), the integers \( \Omega(\gamma; \varphi_2) \) may change: special trajectories can appear/disappear. The changes occur at codimension-1 loci in the space \( \mathcal{B}' \) of quadratic differentials — call these “walls.” (Pictures: examples of 2-3 and 2-\( \infty \) wallcrossing.)

The problem of “wall-crossing” is: given the \( \Omega(\gamma; \varphi_2) \) for one \( \varphi_2 \in \mathcal{B}' \), to determine them at some other \( \varphi_2 \in \mathcal{B}' \).

Kontsevich-Soibelman wrote a remarkable formula, in an \textit{a priori} different context, which turns out to give a complete solution to this problem. The formula involves some surprising-looking ingredients. Let \( A \) be the field of fractions of the group ring \( \mathbb{Z}[\Gamma] \). For any \( \gamma \in \Gamma \), define a formal automorphism \( \mathcal{K}_\gamma \) of \( A \) by

\[ \mathcal{K}_\gamma(\gamma') = \gamma'(1 - \sigma(\gamma)\gamma)^{\langle \gamma : \gamma' \rangle}. \]

Here we had to throw in the annoying object

\[ \sigma : H_1(\Sigma, \mathbb{Z}) \rightarrow \{ \pm 1 \}. \]

A quadratic refinement of the mod 2 pairing. I will not define it
unless someone asks; all we will use of it in what follows is
\[\sigma(\gamma) = \begin{cases} -1 & \text{if there is a saddle conn. with charge } \gamma, \\ +1 & \text{if there is a closed loop with charge } \gamma. \end{cases}\]

Now, we draw a picture: vertical axis \(\vartheta\), horizontal axis any path in \(B'\). On the picture, put a curve \(\ell_\gamma\) for each special trajectory, i.e. for each \(\gamma\) with \(\Omega(\gamma) \neq 0\): \(\ell_\gamma = \{e^{-i\vartheta}Z_\gamma \in \mathbb{R}_-\}\). Now, consider any small “rectangular” paths from \((\vartheta, u)\) to \((\vartheta', u')\) on this picture. Define
\[S(u) = \prod_{\gamma: \vartheta < \arg Z_\gamma < \vartheta'} K_{\gamma}^{\Omega(\gamma; u)}. \quad (5.1)\]

The KSWCF says
\[S(u) = S(u'). \quad (5.2)\]

This equation is strong enough to determine all \(\Omega(\gamma; u')\) given all \(\Omega(\gamma; u)\)!

Examples:
1. If \(\langle \gamma_1, \gamma_2 \rangle = 1\) then
\[K_{\gamma_1} K_{\gamma_2} = K_{\gamma_2} K_{\gamma_1} + K_{\gamma_1}\]
This one governs a situation where two saddle connections join into a third.

2. If \(\langle \gamma_1, \gamma_2 \rangle = 2\) then
\[K_{\gamma_1} K_{\gamma_2} = \left(\prod_{n=1}^{\infty} K_{(n+1)\gamma_2+n\gamma_1}\right) K_{\gamma_1+\gamma_2}^{-2} \left(\prod_{n=\infty}^{1} K_{(n+1)\gamma_1+n\gamma_2}\right) .\]
This one governs a pair of saddle connections joining into a closed loop plus an infinite tower of other saddle connections.

KSWCF as stated also has an evident interpretation in terms of going around closed loops in \((\vartheta, u)\) parameter space.
6 Path lifting

Now, let’s try to explain why KSWCF is true.

We begin by introducing a strange-looking construction: a new “thing you can do with a quadratic differential.”

Fix a pair $(\varphi_2, \vartheta)$. Recall the double cover $\Sigma \rightarrow C$, and the $\mathcal{S}$-walls on $C$.

To every open path $\mathcal{P}$ on $C$, we’ll attach $L(\mathcal{P})$, a formal $\mathbb{Z}$-linear combination of open paths on $\Sigma$, in a way which is “compatible with concatenation”, “twisted homotopy invariant.”

First, suppose $\mathcal{P}$ does not cross any $\mathcal{S}$-walls. In this case, $F(\mathcal{P})$ is the formal sum of the 2 lifts of $\mathcal{P}$ to $\Sigma$:

$$L(\mathcal{P}) = \mathcal{P}^1 + \mathcal{P}^2.$$ 

Next, suppose $\mathcal{P}$ crosses exactly one $\mathcal{S}$-wall, at an intersection point $z$. In this case, $F(\mathcal{P})$ will involve three terms. Two are the naive lifts as before. The third is a path which “takes a detour”. The lift of the $\mathcal{S}$-wall to $\Sigma$ is an open path $S(z)$ running from say $z^1$ to $z^2$. We have

$$L(\mathcal{P}) = \mathcal{P}^1 + \mathcal{P}^2 + \mathcal{P}_+^1 S(z) \mathcal{P}_-^2$$

where the product means concatenation.

Finally, suppose $\mathcal{P}$ is a general path which misses the branch points: then $L(\mathcal{P})$ is constructed by breaking $\mathcal{P}$ into pieces and requiring $L(\mathcal{P})L(\mathcal{P}') = L(\mathcal{P})(\mathcal{P})$ (where we define the product of non-composable paths to be zero).
7 Homotopy invariance

We’d like to ask this to factor through homotopy, but that won’t quite work. You can see that just by considering a closed loop $\mathcal{P}$ around a branch point.

Instead, pass to twisted homotopy: replace smooth paths by their lifts to the unit tangent bundles $\hat{\mathcal{C}}, \hat{\Sigma}$. Identify any path which winds once around the fiber with $-1$. Then, claim: our construction factors through this “twisted homotopy.” (Also, it can be extended to arbitrary paths on $\hat{\mathcal{C}}$, not just ones which arise as lifts of smooth paths on $\mathcal{C}$.)

To check this homotopy property, two illustrative computations:

1. a path which crosses an $\mathcal{S}$-wall twice in opposite directions;
2. a loop around a branch point.

Show one part of the branch point computation: 2 terms cancelling. (NB, it wouldn’t have worked without these detours.)

8 Lifting closed paths

In particular, we can consider $L(\mathcal{P})$ for $\mathcal{P}$ a path beginning and ending at the same $z$. $L(\mathcal{P})$ is a sum of paths beginning and ending at preimages $z^i$, some open, some closed. Define $T(\mathcal{P}) \in A$ as “trace” of $L(\mathcal{P})$: drop open paths, and replace simple closed curves by their homology classes.
9 Morphisms

We’ve defined a rule which assigns to each closed path $\mathcal{P}$ an element $T(\mathcal{P}) \in A$, a formal linear combination of classes in $H_1(\Sigma, \mathbb{Z})$. Now, we may ask: how does $T(\mathcal{P})$ change as we vary $(\varphi_2, \vartheta)$?

For “small” variations which don’t change the topology of the $S$-walls, $T(\mathcal{P})$ does not change (or better, varies continuously, as $\Sigma$ varies). But when the $S$-walls do change topology, $T(\mathcal{P})$ jumps.

Simplest example: two $S$-walls crossing. At the moment when they cross we have a saddle connection, which lifts to some loop $S$. Compare any $L(\mathcal{P})$ before and after the crossing: they differ by a universal transformation, which can be described as an action directly on the paths on $\Sigma$. A path $a$ which crosses $S$ exactly once is split into two pieces $a_1$ and $a_2$ by $S$; after the crossing it is transformed by

$$a = a_1a_2 \mapsto a_1(1 + S)^{\langle a, s \rangle}a_2.$$ 

All $L(\mathcal{P})$ are simply modified by this transformation.

After tracing, this implies that $T(\mathcal{P})$ jumps by

$$\gamma \mapsto \gamma(1 + \gamma S)^{\langle \gamma, \gamma S \rangle}.$$ 

This is exactly the transformation we previously called $K_{\gamma S}$.

More interesting example: a tower of windings collapsing. At the moment of collapse we have a closed trajectory, which again lifts to some loop $S$. Compare any $L(\mathcal{P})$ before and after the crossing: they differ by a universal transformation, which can be described as an action directly on the paths on $\Sigma$. Namely: any path $a$ which crosses $S$ is split into two pieces $a_1$ and $a_2$ by $S$; after the crossing it is transformed by

$$a = a_1a_2 \mapsto a_1(1 - S)^{-\langle a, s \rangle}a_2.$$
Moreover, these closed loops come in pairs. After tracing, this implies that $T(\mathcal{P})$ jumps by

$$
\gamma \mapsto \gamma (1 - \gamma s)^{-2(\gamma, \gamma s)}
$$

This is exactly the transformation we previously called $K_{\gamma s}^{-2}$.

10 Proving the WCF

So far we have produced $T(\mathcal{P}) \in A$ for each path $\mathcal{P}$ on $C$, and shown that as we vary $(\varphi_2, \vartheta)$ along some path in $\mathcal{B}' \times S^1$, all $T(\mathcal{P})$ get transformed by the appropriate product of $K_{\gamma}(\Omega(\gamma))$. If we vary along a closed path then the $T(\mathcal{P})$ must return to themselves. This would prove the desired KSWCF for the $\Omega(\gamma)$, if the $T(\mathcal{P})$ generate the whole $A$.

Indeed the $T(\mathcal{P})$ do generate $A$. (Essentially due to Fock-Goncharov). One way to understand this: “tropicalization” — let the “leading term” $M(\mathcal{P})$ be the $\gamma$ appearing in $T(\mathcal{P})$ with greatest $\text{Re}(e^{i\vartheta Z_\gamma})$. Then show that for any $\gamma \in \Gamma$ there exists a path $\mathcal{P}$ with $M(\mathcal{P}) = \gamma$.

11 Flat connections and Fock-Goncharov coordinates

In trying to understand the WCF we were led to the “path lifting” construction. This construction has other uses: as we will now see it gives a way of relating abelian ($GL(1)$) connections on $\Sigma$ and non-abelian ($GL(2)$) connections on $C \setminus \{z_1, \ldots, z_n\}$.

First, recall a “naive” way of trying to relate the two. Suppose given a complex line bundle $L$ with flat connection on $\Sigma$. The pushforward $E = \pi_* L$ is a rank 2 bundle on $\Sigma$. Does it acquire a flat
connection? Locally, \textit{away from branch points}, $E$ is just the direct sum of 2 line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$, each with a flat connection, so $E$ gets one too: the parallel transport along a path $\mathcal{P}$ is just the sum of the parallel transports along the lifts of $\mathcal{P}$.

But this flat connection in $E$ cannot possibly extend over the branch points: it has \textit{monodromy} (permutation matrix).

Now, our “improved” method. We’ll construct the corrected bundle by building its \textit{sheaf of flat sections}. By definition, a flat section of the improved bundle will be a section of $E$ which is invariant under the improved parallel transport: i.e. under the \textit{abelian} parallel transport along the paths given by $F(\mathcal{P})$. (So it’s discontinuous as a section of $E$, but it will be continuous as a section of the new glued bundle.)

Our homotopy invariance property means this is indeed a (twisted) \textit{flat} connection. (Could get rid of the twisting by choosing spin-structures on $C$ and $\Sigma$, but let’s not.) So we get a “non-abelianization” map $\nabla^{ab} \mapsto \nabla$, from the moduli space of flat $GL(1)$-connections over $\Sigma$ to the moduli space $\mathcal{M}$ of flat $GL(2)$-connections over $C$. (More precisely, flat $GL(2)$-connections over $C$ with the extra data of a flag at each puncture.)

In this picture $T(\mathcal{P})$ has a particularly concrete meaning: it is giving the trace of the holonomy of $\nabla$ around $\mathcal{P}$, as a function of the holonomies $\mathcal{X}_\gamma$ of $\nabla^{ab}$ around loops $\gamma$ in $\Sigma$.

Do we get all $GL(2)$-connections $\nabla$ this way? Almost: this “non-abelianization” map is actually an isomorphism onto an open dense patch of $\mathcal{M}$. This is basically a result of Fock-Goncharov: strictly speaking they studied $SL(2)$-connections, but the overall $GL(1)$ part goes through trivially (I hope; could be some $\mathbb{Z}_2$ subtleties here to
fuss with).

Their proof goes by constructing the explicit inverse of our map: “abelianization.” Since a $GL(1)$-connection is specified by its $\mathbb{C}^\times$-valued holonomies, concretely this amounts to specifying an open dense coordinate patch on the space of flat $GL(2)$-connections. The $SL(2)$ part is the interesting part. Fock-Goncharov build these coordinates by taking cross-ratios of flat sections. (Picture.)

More precisely, this is one coordinate system for every $S$-wall network; when the $S$-wall network changes topology, the coordinate system jumps. The different coordinate systems are related by “cluster transformations”: a concrete instantiation of the $K_\gamma$ we wrote before, now acting on actual functions rather than formal variables. Quite interesting structure, for reasons I’m not fully competent to explain.

12 Higher rank

The story seems to have a natural generalization to “higher rank.”

Starting point: replace the quadratic differential $\varphi_2$ by a tuple $(\varphi_2, \ldots, \varphi_K)$ where $\varphi_i$ is a section of $K^i$.

The special trajectories we studied before could be understood as loci where the $S$-wall network jumped. So: what is the appropriate generalization of the $S$-walls here?

As before, we can define a spectral curve by

$$\Sigma = \{ \lambda^K + \sum_{n=2}^{K} \varphi_n \lambda^{K-n} = 0 \} \subset T^*C.$$  \hspace{1cm} (12.1)

A $K$-fold cover of $\Sigma$. 
For any choice of a labeling of sheets of $\Sigma$ (locally defined), we thus have $K$ 1-forms on $C$, $\lambda_1, \ldots, \lambda_K$. We define an $ij$-trajectory to be one along which the 1-form $\lambda_i - \lambda_j$ is real (and positive). Our $S$-wall network will be built out of these $ij$-trajectories.

Moreover, using our $S$-wall network we want to be able to build a path-lifting rule, with the same kind of twisted homotopy invariance as we had in the $K = 2$ case.

Branch points are labeled by transpositions $(ij)$. To get the homotopy invariance around each $(ij)$ branch point, we will need to have 3 $S$-walls emerging. (Draw the picture.) But now we have a new problem: the $S$-walls might collide. Suppose an $(ij)$ and a $(jk)$ $S$-wall collide. In this case we will have failure of homotopy invariance (a loop around the collision point is not equivalent to a trivial one). The way to fix it is to add a new $(ik)$ $S$-wall emerging from the branch point. This new $S$-wall then evolves along with the rest. We build up a rather complicated, but controlled, structure. (NB, it is also possible for $S$-walls to die.) If there are punctures, with each $\varphi_i$ having a pole of order $i$, then all $S$-walls eventually wind up at the punctures.

Using this new $S$-wall network we can define a path-lifting rule $\mathcal{P} \mapsto L(\mathcal{P})$, the straightforward generalization of what we did in the $K = 2$ case; take traces to get $T(\mathcal{P})$. As before, the crucial question is: when does $T(\mathcal{P})$ jump discontinuously? Answer: whenever two $S$-walls collide head-to-head.

The most obvious way for this to happen is to have a saddle connection, like before. But there are also more interesting possibilities. (Show examples.) Whenever the $S$-walls collide, there is a corresponding finite subnetwork. Its lift to $\Sigma$ defines a charge
\[ \gamma \in \Gamma = H_1(\Sigma, \mathbb{Z}). \]

The analysis of \( T(\mathcal{P}) \) at the special loci \( e^{-i\theta} Z_\gamma \in \mathbb{R}_- \) goes much like before: they jump by an automorphism \( \mathcal{K}_c^\gamma \) where \( c \) depends on the topology of the subnetwork. Simple examples: three-pronged network gives \( c = +1 \), loop with attached edge gives \( -1 \). Conjecture: every network gives \( \pm 1 \). At any rate, it’s in principle straightforward to determine the contribution from any particular network. So we will obtain invariants \( \Omega(\gamma) \) like before; and the same argument we used would be expected to prove KSWCF in this setting too (if there are “enough” \( T(\mathcal{P}) \).)

All the usual questions about special trajectories of quadratic differentials should be interesting for these finite subnetworks, too. (e.g. how many of them with length \( \leq L \)?)

Our path-lifting construction leads to “non-abelianization” map relating \( GL(1) \)-connections on \( \Sigma \) to \( GL(K) \)-connections on \( C \). Conjecture: as before, this map is onto an open dense subset of the moduli space \( \mathcal{M} \) of such connections. So each \( S \)-wall network would give a set of “Fock-Goncharov-like” coordinates on \( \mathcal{M} \). If we take \( \varphi_3, \ldots, \varphi_K \) to be very small and arranged in a particular way, we can actually identify them with the honest Fock-Goncharov coordinates for higher rank. (Show picture of spin-lift and the higher-rank flip.)

13 WKB

There is another well-known approach to “abelianizing” a connection, or more precisely a family of connections: WKB. Suppose given a family of \( GL(K) \)-connections of the form

\[ \nabla = \varphi/\zeta + D(\zeta) \quad (13.1) \]
where $\varphi$ is a $gl(K)$-valued matrix and $D$ a connection, regular at $\zeta = 0$. One often wants (e.g. in quantum mechanics) to study the flat sections ($\nabla \psi = 0$) in the limit $\zeta \to 0$. WKB approximation says: just diagonalize $\varphi$,

$$\varphi = \text{diag}(\lambda_i) \quad (13.2)$$

and then construct formal solutions in the form

$$\psi^{WKB}_i = \exp \left[ \int \frac{\lambda_i}{\zeta} \right] e_i(\zeta) \quad (13.3)$$

where $e_i(\zeta)$ is a power series in $\zeta$, determined by iteratively plugging into the flatness equation. The $\psi^{WKB}_i(\zeta)$ then look like they define an abelian connection over $\Sigma$ of the form

$$\nabla^{ab,WKB} = \frac{\lambda}{\zeta} + D^{ab,WKB}(\zeta) \quad (13.4)$$

whose pushforward would be $\nabla$.

But as we know, you can’t really construct $\nabla$ this way (if you could, $\nabla$ would have monodromy around branch points). So what goes wrong? The point is that the series defining $\psi^{WKB}_i$ typically is not a convergent series: it only allows us to abelianize the connection in a formal neighborhood of $\zeta = 0$.

14 Comparing our story with WKB

We constructed a “de-abelianization” map, using the additional datum of a pair $(\vartheta, \varphi_2, \ldots, \varphi_K)$. Conjecture (true for $K = 2$): it’s invertible, so gives “abelianization” map (defined on dense open subset).
Now suppose as above that $\nabla = \varphi/\zeta + D(\zeta)$, and take $\varphi$ to be the coefficients of the characteristic polynomial of $\varphi$. Apply the abelianization map.

This in particular provides actual flat sections $\psi_i$ on the complement of the $S$-walls. (Concretely, for $K = 2$, the exponentially-smaller monodromy eigensections at the “nearest” puncture.) These $\psi_i$ jump at the $S$-walls.

Conjecture (true for $K = 2$): this construction is compatible with the WKB method, in the sense that the actual flat sections $\psi_i$ have asymptotic expansion given by $\psi_i^{WKB}$, as $\zeta \to 0$. (Although they are not continuous!)

This WKB property is vital for some applications. It wouldn’t have worked if we chose a “random” network; depends on using the network that’s really defined by $(\varphi_2, \ldots, \varphi_K)$.

The jumps of $\psi_i^{WKB}$ are related to “WKB connection formula.” So this is a re-telling of a somehow familiar story (Ecalle, Voros etc.) The part involving closed geodesics may be new, also the higher rank story.

15 Hyperkahler metrics

One application of this WKB analysis is a new way of thinking about the Hitchin system.

An amazing fact [Hitchin, Simpson, Corlette, Donaldson]. Consider the space $\mathcal{M}$ of flat $GL(K, \mathbb{C})$-connections. Given any $\nabla$ (subject to some “stability” condition, automatically satisfied in our case with generic punctures) you can find a decomposition

$$\nabla = \varphi + D + \bar{\varphi}, \quad \text{(15.1)}$$
where $D$ is unitary and $\varphi$ is adjoint of $\varphi$ (with respect to some metric), and we have

$$F_D + [\varphi, \bar{\varphi}] = 0, \quad (15.2)$$
$$\bar{D}\varphi = 0. \quad (15.3)$$

So, now, let’s try starting from the pair $(D, \varphi)$. We can build $\nabla$ from them, but in fact we can build a 1-parameter family of flat connections:

$$\nabla^{(\zeta)} = \varphi/\zeta + D + \bar{\varphi}\zeta. \quad (15.4)$$

So $\mathcal{M}$ is identified with the complex manifold of flat $GL(K, \mathbb{C})$-connections in many different ways. i.e. $\mathcal{M}$ has many different complex structures $J^{(\zeta)}$, $\zeta \in \mathbb{C}^\times$. In particular, if you fix $J_1 = J^{(\zeta=1)}$, $J_2 = J^{(\zeta=i)}$, $J_3 = J^{(\zeta=0)}$, then $J_1 J_2 = J_3$ and cyclic permutations: quaternion algebra.

$\mathcal{M}$ also has a holomorphic symplectic form: defined in terms of symplectic quotient starting from

$$\varpi^{(\zeta)} = \int_C \text{Tr} \delta \nabla^{(\zeta)} \wedge \delta \nabla^{(\zeta)}. \quad (15.5)$$

Expands explicitly as

$$\varpi^{(\zeta)} = \frac{\omega_1 + i\omega_2}{\zeta} + \omega_3 + (\omega_1 - i\omega_2)\zeta \quad (15.6)$$

where $\omega_1$, $\omega_2$, $\omega_3$ are three real symplectic forms. In fact, they are Kähler forms with respect to the three complex structures $J_1$, $J_2$, $J_3$, determining a single Riemannian metric $g$. $g$ is thus called a hyperkähler metric. In particular it’s Ricci-flat.

Now, suppose we want to actually construct this metric in some concrete terms. It’s enough to construct $\varpi^{(\zeta)}$. But, the symplectic
structure looks rather complicated. Good news: the abelianization map we have discussed is actually also a symplectomorphism! And the symplectic structure on the space of $GL(1)$-connections is very simple: just $\langle d\log X, d\log X \rangle$.

Why does this help? After all, $X$ is still a complicated function on the original $M$. But, we know a lot about it. WKB determines its leading asymptotic as $\zeta \to 0, \infty$ as $X \sim \exp[Z_\gamma/\zeta], X \sim \exp[\bar{Z}_\gamma \zeta]$ respectively. And we know where its discontinuities are. Thus we have a Riemann-Hilbert problem which can be solved by an explicit integral equation.

That gives a recipe for constructing the actual $\varpi$ and hence the hyperkähler metric $g$.

16 Some physics

Now, what is the meaning of all this for physicists?

In many (all?) cases, DT invariants can be understood in terms of 4-dimensional supersymmetric quantum field theory (QFT). I won’t say what a QFT is: suffice to say that it is supposed to have a Hilbert space $\mathcal{H}$, with subspace $\mathcal{H}^1$ (space of “1-particle states”) which forms a unitary representation of a super extension of the Poincare group $ISO(3,1)$.

$ISO(3,1)$ has a Casimir operator “$M^2$” which, acting on $\mathcal{H}_1$, tells us the mass-squared of the particles. Our extension also has a second Casimir operator $Z$, complex-valued even in unitary representations. All unitary representations have $M \geq |Z|$. Moreover the representations with $M = |Z|$ are special (“short”). States in these representations are called “BPS states.” There is an index $\Omega$ which
counts the multiplicity of such representations, which cannot change under continuous deformations of the representation $\mathcal{H}^1$ (designed so that it vanishes for “long” representations).

The theories we consider are actually (in the IR) abelian gauge theories. Like electromagnetism. In such a theory, $\mathcal{H}$ has a decomposition into “charge sectors” labeled by a lattice $\Gamma$ of electromagnetic charges,

$$\mathcal{H} = \bigoplus_{\gamma} \mathcal{H}_\gamma.$$  

Moreover the Casimir operator $Z$ acts as a scalar $Z_\gamma$ in each sector.

We can compute the index $\Omega$ in each $\mathcal{H}_\gamma$ separately: get $\Omega(\gamma) \in \mathbb{Z}$. Now, we may ask, what happens when we vary the parameters of the theory? For small variation, just get some small variation of each $\mathcal{H}_\gamma^1$, so $\Omega(\gamma)$ is invariant. But exactly when different $Z_\gamma$ become aligned, $\mathcal{H}_\gamma^1$ actually “mixes with the continuum” and $\Omega(\gamma)$ becomes ill-defined. Thus we have the possibility of wall-crossing. Indeed, there is a nice semiclassical picture of a “bound state” which decays [Denef].

How to get such an $\mathcal{N} = 2$ supersymmetric QFT? One attractive option: string theory on Calabi-Yau threefold. IIB: BPS states come from D3-branes wrapped around special Lagrangian 3-cycles. But there is also a second way of getting an $\mathcal{N} = 2$ supersymmetric QFT. Namely, there is a somewhat mysterious 6-dimensional QFT (“theory $\mathcal{X}$”), or more precisely one $\mathcal{X}_g$ for each ADE algebra $g$ (plus more trivial abelian ones). Formulating this theory where we take spacetime to be $C \times \mathbb{R}^{3,1}$ we get the theory we have been (implicitly) discussing.

The two pictures are not unrelated: one way of realizing theory
$\chi$ is as the Type IIB string theory near an ADE singularity. Then by wrapping D3-branes around the collapsing 2-cycles of the ADE singularity, we get effective “strings.”