Wall-Crossing (3)

Last time: WCF for any \(N=2, d=4\) field theory.

Now let’s consider some theories when BPS states can be understood geometrically.

The idea: There is a rather mysterious SCFT in \(d=6\) with \((2,0)\) SUSY. More exactly, a family of them: one for each ADE Dynkin diagram. No known Lagrangian description: only get info rather indirectly!

Construction from M-theory: (for \(A_{K-1}\) theories)

In M-theory, have the \(M5\)-brane. Consider \(K\) \(M5\)-branes, look at the low energy dynamics.

The theory has a "Conifold branch" when the \(K\) branes are separated in the transverse \(R^5\).

If \(K>2\), many different flavors of strings:

Compactify the \((2,0)\) \(A_K\) theory on a Riemann surface \(C\) (carrying "defect operators" inserted at points).
With an appropriate topological twist, this leads to an $N=2$ theory in 4 dimensions.

In M-theory, one think of this as studying:

\[ M\text{-theory in } \mathbb{T}^*C \times \mathbb{R}^6,1 \]

\[ \text{M5-branes on } C \times \mathbb{R}^3,1 \]

To go out on Coulomb branch, try to separate the M5-branes:

Best we can do is separate $K$ branes on $C \to$ 1 brane on $\Sigma$, a $K$-fold cover of $C$.

$\Sigma$ is given by an equation of the form

\[ \lambda^K + \psi_2 \lambda^{K-2} + \psi_3 \lambda^{K-3} + \ldots + \psi_K = 0 \quad \lambda \in \mathbb{T}^*C \]

\(\psi_n\) is a meromorphic n-differential on $C$.

Different choices of the $\psi_n \leftrightarrow$ different "shapes" of $\Sigma \\
\leftrightarrow$ different points of Coulomb branch $\mathcal{B}$. 
$E_x \ K=2$

$C = \mathbb{CP}^2 \ w/2 \ punctures$

$\Sigma$ given by $\lambda^2 = \mathcal{U}_2$

where $\mathcal{U}_2(z) = \left( \frac{\lambda^2}{z} + 2u + \Lambda^2 \right) \left( \frac{d_z}{z} \right)^2$

So, in local coordinates ($\lambda = x \ d_z$)

$\Sigma'$ is $\{ \chi^2 = \frac{\lambda^2}{z^2} + \frac{2u}{z^2} + \frac{A^2}{z} \} \subset C^1 \times C^1$

The $N=2$ theory that we get from this construction is the pure $N=2$ SYM with gauge group $G = SU(2)$!

And $\Sigma'$ is the "Seiberg-Witten curve."

BPS state: $M_2$-branes stretched between the sheets of $\Sigma'$.

Tension depends on $\lambda$: it's ~ the distance between brane, $|\lambda| = |x \ d_z|$

So, the total mass of the state from 4d Pov is

$M = \int |\lambda| = \int |x \ d_z|$

while the central charge is

$\mathcal{Z}_\chi = \oint \lambda = \oint x \ d_z$
BPS bound: \( M \geq |2| \)

Here just says \( \int |x \cdot d\Sigma| \geq \int |x \cdot d\Sigma| \)

So the string gives a BPS state only if \( M = |2| \) i.e. \( |\int x \cdot d\Sigma| = \int |x \cdot d\Sigma| \)

i.e. \( x \cdot d\Sigma \) has constant phase along the string.

To make a BPS state, we need a string w/ finite total mass.

Two ways to get one:

- BPS hypermultiplet
  \((S_{\Omega}, \Omega = +1)\)

- BPS vector multiplet
  \((S_{\frac{1}{2}}, \Omega = -2)\)

The boundary of the M2-brane determines a cycle \( Y \) on \( \Sigma \)

\[ Z_Y = \oint_Y \lambda \quad \text{- Seiberg-Witten formula} \]

In higher-K theories we expect more complicated possibilities.
$K=3$

So the appearance/dis. of these objects
as we vary module should be governed by the WCF...