M 340L Fall 2010 (55430), Midterm 1

Answer the questions on the scratch paper provided. Box your answers. Correct answers will always get full credit even if no work is shown; if you want partial credit, show your work. All scratch paper will be collected at the end.

Name: ____

Question	Points	Score
1	20	
2	15	
3	10	
4	20	
5	15	
6	30	
7	5	
Total:	115	

Problem 1 (20 points).

Consider the matrix $A = \begin{bmatrix} -3 & -4 & 6 \\ 1 & 2 & 0 \\ 0 & 2 & 6 \end{bmatrix}$.

(a) (8 points) Find all solutions of $A\mathbf{x} = \begin{bmatrix} -1 \\ 3 \\ 8 \end{bmatrix}$. If the solution set is nonempty, describe it in parametric form. (Recall that "parametric form" means giving a description of the solution set of the form $\mathbf{x} = t\mathbf{v} + \mathbf{p}$, or $\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \mathbf{p}$.)

Solution: Build the augmented matrix $\begin{bmatrix} -3 & -4 & 6 & -1 \\ 1 & 2 & 0 & 3 \\ 0 & 2 & 6 & 8 \end{bmatrix}$. Reduce it to reduced row echelon form, $\begin{bmatrix} 1 & 0 & -6 & -5 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. This amounts to reducing the equations to $x_1 = -5 + 6x_3$, $x_2 = 4 - 3x_3$, with x_3 a free variable. In parametric form, writing x_3 as t, this is $\mathbf{x} = t\begin{bmatrix} 6 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} -5 \\ 4 \\ 0 \end{bmatrix}$.

(b) (8 points) Find all solutions of $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. If the solution set is nonempty, describe it in parametric form.

Solution: Build the augmented matrix $\begin{bmatrix} -3 & -4 & 6 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 2 & 6 & 0 \end{bmatrix}$. Reducing it to row echelon form you see it has a pivot in the last column. (The reduced row echelon form is $\begin{bmatrix} 1 & 0 & -6 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, although you don't have to go all the way to the reduced row echelon form to see where the pivots are.) So the system is inconsistent: there are no solutions.

(c) (4 points) Do the columns of A span $\mathbb{R}^3$? If not, name a vector in $\mathbb{R}^3$ that is not in the span of the columns of A.

Solution: A vector $\mathbf{b}$ is in the span of the columns of A if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution. In part b we saw that this equation does not have a solution when $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. So the columns of A do not span $\mathbb{R}^3$, and the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not in the span of the columns of A.

Problem 2 (15 points).

Suppose

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

(a) (5 points) Is the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ linearly independent? If it is not, give a nontrivial solution of the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{0}$.

Solution: Yes, this set is linearly independent: it is a set of two vectors, neither of which is a scalar multiple of the other.

(b) (5 points) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? If it is not, give a nontrivial solution of the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$.

Solution: They are linearly dependent, because $\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$, i.e. there is a nontrivial solution of $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ given by $x_1 = 1$, $x_2 = -2$, $x_3 = 1$. You could find this solution either by a trick I showed on the practice problems, or by the usual method of row-reducing the augmented metric $\begin{bmatrix} 1 & 4 & 7 & 0 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 0 \end{bmatrix}$.

(c) (5 points) Is $\mathbf{v}_3$ in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$? If so, write $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Solution: We just showed that $\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$. Now rearrange this to make $\mathbf{v}_3 = 2\mathbf{v}_2 - \mathbf{v}_1$. That shows that $\mathbf{v}_3$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$, and gives $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Problem 3 (10 points).

(a) (2 points) Draw a picture of two vectors $\mathbf{v}_1$, $\mathbf{v}_2$ in $\mathbb{R}^2$ such that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

Solution: Two vectors in a plane pointing in different directions.

(b) (2 points) Draw a picture of two vectors $\mathbf{v}_1$, $\mathbf{v}_2$ in $\mathbb{R}^2$ such that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent.

Solution: Two vectors in a plane pointing in the same direction, or in exactly opposite directions (so one is a scalar multiple of the other).

(c) (3 points) Draw a picture of three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ in $\mathbb{R}^3$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

Solution: Three vectors in space which are not in any common plane (the simplest is to just take unit vectors pointing along the three coordinate axes).

(d) (3 points) Draw a picture of three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ in $\mathbb{R}^3$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent but each of the sets $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, $\{\mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

Solution: Three vectors lying in a plane, such that no two point in the same direction, and no two point in exactly opposite directions (so no two are scalar multiples of one another).

Problem 4 (20 points).

Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$, given by the formula

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 3x_1 + 3x_2 \\ 0 \end{bmatrix}.$$

(a) (5 points) Find the standard matrix of T, i.e. the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

Solution: There are a few ways to get the answer. One is to say that since A maps $\mathbb{R}^2$ to $\mathbb{R}^3$ it must be a 3×2 matrix, then write such a matrix with undetermined entries, i.e. $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$ and figure out what the a_{ij} have to be just by using the row-column rule to calculate $A\mathbf{x}$. For example, the first entry of $A\mathbf{x}$ is $a_{11}x_1 + a_{12}x_2$, and we want that to equal $x_1 + x_2$, for all $\mathbf{x}$; that means $a_{11} = 1$ and $a_{12} = 1$. Another way to get the answer is to use the fact that $A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix}$ where $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Either way, in the end you get $A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \\ 0 & 0 \end{bmatrix}$.

(b) (5 points) Is T a 1-1 map?

Solution: The matrix A does not have pivots in every column, so T is not 1-1. (Or: $T(\mathbf{e}_1) = T(\mathbf{e}_2)$, so T is not 1-1.)

Suppose U is another linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, given by the formula

$$U\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 4x_2 \\ 2x_1 \end{bmatrix}.$$

(c) (2 points) Find the standard matrix of U, i.e. the matrix B such that $U(\mathbf{x}) = B\mathbf{x}$.

Solution: This is much like part a; using either of the methods from there, the answer here is $B = \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}$.

(d) (4 points) Find the matrix C such that $T(U(\mathbf{x})) = C\mathbf{x}$.

Solution: The composition of linear transformation corresponds to multiplication of matrices: $T(U(\mathbf{x})) = T(B\mathbf{x}) = A(B\mathbf{x}) = (AB)\mathbf{x}$, so the matrix C we want is just $C = AB$. Doing that multiplication using A and B from the previous parts, you get $C = \begin{bmatrix} 2 & 4 \\ 6 & 12 \\ 0 & 0 \end{bmatrix}$.

(e) (4 points) Find the matrix D such that $U(U(\mathbf{x})) = D\mathbf{x}$.

Solution: Much like part d; $D = BB = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$.

Problem 5 (15 points).

Consider the matrix $A = \begin{bmatrix} 12 & -5 \\ -7 & 3 \end{bmatrix}$.

(a) (5 points) Does A have an inverse? If so, find it.

Solution: Using the formula for inverse of a 2×2 matrix, $A^{-1} = \begin{bmatrix} 3 & 5 \\ 7 & 12 \end{bmatrix}$.

The fastest way to solve the next two parts is to use your answer from the first part.

(b) (5 points) Find a vector $\mathbf{x}$ such that $A\mathbf{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, or write "none" if there is no such $\mathbf{x}$.

Solution: $\mathbf{x} = A^{-1}\begin{bmatrix}-1\\1\end{bmatrix}$ does the job (to see why it works, try multiplying both sides by A from the left). Calculating that product using the answer from part a, $\mathbf{x} = \begin{bmatrix}2\\5\end{bmatrix}$.

(c) (5 points) Find a matrix B such that $AB = \begin{bmatrix}1 & 0 & 1\\0 & 2 & 1\end{bmatrix}$, or write "none" if there is no such B.

Solution: $B = A^{-1}\begin{bmatrix}1 & 0 & 1\\0 & 2 & 1\end{bmatrix}$ does the job (to see why it works, try multiplying both sides by A from the left). Calculating that product using the answer from part a, $B = \begin{bmatrix}3 & 10 & 8\\7 & 24 & 19\end{bmatrix}$.

Problem 6 (30 points).

True or False. If a statement is sometimes true and sometimes false, write "false". You do not have to justify your answers. There will be no partial credit for this problem.

(a) (3 points) If A is a 5×4 matrix and the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution for $\mathbf{x}$, then the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is 1-1.

Solution: TRUE. Any linear transformation T for which $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution is 1-1.

(b) (3 points) If A is a 4×6 matrix, then the equation $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

Solution: TRUE. First, a homogeneous linear system is always consistent, so the equation has at least one solution. It will have infinitely many solutions if and only if there are free variables. This matrix can have at most 4 pivots, but it has 6 columns, so there are at least 2 columns without pivots, i.e. there are at least 2 free variables.

(c) (3 points) If A is a 4×6 matrix, then the equation $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions for every nonzero $\mathbf{b}$.

Solution: FALSE. The system might be inconsistent for some $\mathbf{b}$. (It is true that *if the system is consistent* then it has infinitely many solutions, since there are 2 free variables, like part b.)

(d) (3 points) If A and B are 3×3 matrices, then $AB = BA$.

Solution: FALSE. Matrices generally do not have to commute.

(e) (3 points) If A and B are diagonal matrices, and B is invertible, then $B^{-1}AB = A$.

Solution: TRUE. Diagonal matrices do commute. So $B^{-1}AB = B^{-1}BA = IA = A$.

(f) (3 points) The transpose of a product of matrices is $(AB)^T = B^T A^T$.

Solution: TRUE.

(g) (3 points) If A is a 3×5 matrix with 3 pivots, then the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ has image equal to $\mathbb{R}^3$.

Solution: TRUE. A has a pivot in every row, which is equivalent to saying $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$, which is equivalent to saying $T(\mathbf{x}) = A\mathbf{x}$ has image $\mathbb{R}^3$.

(h) (3 points) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a linearly independent set of vectors in $\mathbb{R}^4$, then $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is $\mathbb{R}^4$.

Solution: TRUE. These are two of the long list of statements that are all equivalent to saying the matrix $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix}$ is invertible. (Or: linear independence of the columns means the matrix has a pivot in every column, but since it is 4×4 this also means it has a pivot in every row, which means the columns span $\mathbb{R}^4$.)

(i) (3 points) If A and B are invertible 4×4 matrices, then $A + B$ is also an invertible 4×4 matrix.

Solution: FALSE. For example, if A is invertible, then $B = -A$ is also invertible, but $A + B$ is the zero matrix, which is definitely not invertible.

(j) (3 points) If $\mathbf{v}$ is any vector in $\mathbb{R}^3$, $\text{Span}\{\mathbf{v}, 2\mathbf{v}, 3\mathbf{v}\} = \text{Span}\{\mathbf{v}\}$.

Solution: TRUE. Any vector which is a linear combination of $\mathbf{v}$, $2\mathbf{v}$ and $3\mathbf{v}$ is a scalar multiple of $\mathbf{v}$, and vice versa.

Problem 7 (5 points).

Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ and $\mathbf{u}$, $\mathbf{v}$ are two vectors, such that $\{\mathbf{u}, \mathbf{v}\}$ is a linearly independent set and

$$T(\mathbf{u}) = \mathbf{v}, \qquad T(\mathbf{v}) = \mathbf{u}.$$

(a) (3 points) Find a nonzero vector $\mathbf{w}$ such that $T(\mathbf{w}) = \mathbf{w}$.

> **Solution:** If we take $\mathbf{w} = \mathbf{u} + \mathbf{v}$ then $T(\mathbf{w}) = T(\mathbf{u}) + T(\mathbf{v}) = \mathbf{v} + \mathbf{u} = \mathbf{w}$ as desired.

(b) (2 points) Find the matrix A such that $T(T(\mathbf{x})) = A\mathbf{x}$.

> **Solution:** $T(T(\mathbf{u})) = \mathbf{u}$ and $T(T(\mathbf{v})) = \mathbf{v}$. And any vector $\mathbf{x}$ can be written as a linear combination of $\mathbf{u}$ and $\mathbf{v}$ (this is the same statement that appeared in problem 6i, except that there we considered $\mathbb{R}^4$ and now we have $\mathbb{R}^2$). From this you can see that $T(T(\mathbf{x})) = \mathbf{x}$ for every $\mathbf{x}$. In other words, T is the identity operation, whose standard matrix is the identity matrix, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.