M340L Final Exam Solutions, May 13, 2010

1. *(10 points) The matrix* $A = \begin{pmatrix} 1 & 1 & 3 & 1 & 7 & 7 \\ 1 & 2 & 5 & 3 & 20 & 16 \\ 2 & 4 & 10 & 7 & 45 & 36 \end{pmatrix}$ *is row-equivalent to* $\begin{pmatrix} 1 & 0 & 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 5 & 4 \end{pmatrix}$.

a) Are the vectors $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$, *and* $\begin{pmatrix} 3 \\ 5 \\ 10 \end{pmatrix}$ *linearly independent?*

No. $\begin{pmatrix} 3 \\ 5 \\ 10 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$. This can be seen from the RREF form, as the 3rd column equals the first plus twice the second. Alternately, the RREF form of the first three columns doesn't have a pivot in each column.

By itself, the fact that these aren't pivot columns doesn't mean they can't be linearly independent. For instance, the third, firth and sixth columns are linearly independent even though none of them are pivot vectors.

b) Do the vectors $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 5 \\ 10 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}$, $\begin{pmatrix} 7 \\ 20 \\ 45 \end{pmatrix}$ *and* $\begin{pmatrix} 7 \\ 16 \\ 36 \end{pmatrix}$ *span* $\mathbf{R}^3$*?*

Yes, since the RREF form has pivots in all rows.

c) Find bases for the column space of A, for the row space of A, and for the null space of A.

For the column space, use the pivot columns, namely $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}$.

For the row space, use the nonzero rows of the RREF, namely $(1, 0, 1, 0, -1, 2)$, $(0, 1, 2, 0, 3, 1)$, $(0, 0, 0, 1, 5, 4)$.

For the null space, rewrite the equations $A_{rref}\mathbf{x} = 0$ to get the basis $\begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -3 \\ 0 \\ -5 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -2 \\ -1 \\ 0 \\ -4 \\ 0 \\ 1 \end{pmatrix}$.

2 (15 points) Let $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ *be a basis for* $\mathbf{R}^3$*, and let* $\mathcal{E}$ *be the standard basis.*

a. Compute the change-of-basis matrices $P_{\mathcal{E}\mathcal{B}}$ *and* $P_{\mathcal{B}\mathcal{E}}$*.*

$$P_{\mathcal{E}\mathcal{B}} = (\mathbf{b}_1 \mathbf{b}_2 \mathbf{b}_3) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \ P_{\mathcal{B}\mathcal{E}} = P_{\mathcal{E}\mathcal{B}}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

b. If $\mathbf{x} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$*, what is* $[\mathbf{x}]_{\mathcal{B}}$*?*

$$[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}}[\mathbf{x}]_{\mathcal{E}} = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}.$$

c. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ *be a linear transformation given by the formula* $T(\mathbf{x}) = \begin{pmatrix} 3x_1 + 2x_2 + x_3 \\ 2x_1 - x_3 \\ x_2 \end{pmatrix}$*. Find the standard matrix of* T *(relative to the standard basis).*

$$[T]_{\mathcal{E}} = (T(\mathbf{e}_1)T(\mathbf{e}_2)T(\mathbf{e}_3)) = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

d. Find the matrix of T *relative to the* $\mathcal{B}$ *basis.*

$[T]_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}}[T]_{\mathcal{E}}P_{\mathcal{E}\mathcal{B}} = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. Note that you have to multiply by $P_{\mathcal{B}\mathcal{E}}$ on one side and $P)\mathcal{E}\mathcal{B}$ on the other, unlike the change-of-basis formula for vectors, where you only multiply on one side.

3. (12 points)

For each of these square matrices, either find the inverse or explain why the inverse does not exist.

(a) $\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$

Since the determinant is 2, the inverse is $\frac{1}{2}\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 3 & 5 \\ 2 & 1 & 4 \\ 3 & 4 & 9 \end{pmatrix}$

The inverse does not exist, since the matrix only has rank 2. (The third row is the sum of the first two, so row reduction quickly leads to a row of zeroes.)

c) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & 4 & 5 \end{pmatrix}$

By row reducing $[A|I]$ to $[I|A^{-1}]$ we get $A^{-1} = \begin{pmatrix} -13 & 4 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

4. (10 points) *(a) Write down the characteristic equation of the matrix* $A = \begin{pmatrix} 3 & 7 \\ 1 & 4 \end{pmatrix}$*. You do not need to find the eigenvalues or eigenvectors.*

$\det \begin{pmatrix} 3-\lambda & 7 \\ 1 & 4-\lambda \end{pmatrix} = \lambda^2 - 7\lambda + 5$, so our characteristic equation is $\lambda^2 - 7\lambda + 5 = 0$. Note that "$\lambda^2 - 7\lambda + 5$," by itself, isn't an equation.

b) The eigenvalues of $B = \begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix}$ *are 3 and -3. Find bases for* E_3 *and* E_{-3}*. Is* B *diagonalizable?*

In each case we row-reduce $A - \lambda I$. For $\lambda = 3$ this yields $\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$, hence the eigenvector $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$. For $\lambda = -3$ this yields $\begin{pmatrix} 1 & 1/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, hence the TWO eigenvectors $\begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix}$.

Since there are three linearly independent eigenvectors (one for 3 and two for -3), the matrix is diagonalizable. There are only two *eigenvalues*, but there are three *eigenvectors*, which is what counts. This is the flip side to a problem on the second midterm.

5. (15 points) The matrix $\begin{pmatrix} 3 & 3 \\ 3 & -5 \end{pmatrix}$ *has eigenvalues* $\lambda_1 = 4$ *and* $\lambda_2 = -6$ *and corresponding eigenvectors* $\mathbf{b}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ *and* $\mathbf{b}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

a. Find the coordinates of $\begin{pmatrix} 13 \\ 1 \end{pmatrix}$ *in the* $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ *basis.*

Note that the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ are orthogonal, which makes decomposing easy. $\mathbf{x} = \frac{\mathbf{x}\cdot\mathbf{b}_1}{\mathbf{b}_1\cdot\mathbf{b}_1}\mathbf{b}_1 + \frac{\mathbf{x}\cdot\mathbf{b}_2}{\mathbf{b}_2\cdot\mathbf{b}_2}\mathbf{b}_2 = \frac{40}{10}\mathbf{b}_1 + \frac{10}{10}\mathbf{b}_2$, so $[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

You can also do this by row-reducing $\left(\begin{array}{cc|c} 3 & 1 & 13 \\ 1 & -3 & 1 \end{array}\right)$ or computing $\begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix}^{-1} \begin{pmatrix} 13 \\ 1 \end{pmatrix}$, but a lot of people who tried to solve the problem these ways made numerical errors or forgot to divide by the determinant in the formula for $\begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix}^{-1}$.

b. If $\mathbf{x}(n+1) = A\mathbf{x}(n)$ *and* $\mathbf{x}(0) = \begin{pmatrix} 13 \\ 1 \end{pmatrix}$, *find* $\mathbf{x}(n)$ *for all* n. *What is the dominant eigenvector (and eigenvalue) for this problem?*

Since $\mathbf{y}(0) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $\mathbf{y}(n) = \begin{pmatrix} 4^{n+1} \\ (-6)^n \end{pmatrix}$ and $\mathbf{x}(n) = 4^{n+1}\mathbf{b}_1 + (-6)^n\mathbf{b}_2$. Since $|-6| > |4|$, the dominant eigenvector is $\mathbf{b}_2$, and the dominant eigenvalue is -6.

The fact that -6 is negative is irrelevant. $(-6)^n$ is still a lot bigger (in magnitude) than 4^n for n large, so the second term is a lot bigger than the first. That's what "dominant" means.

c. Suppose instead that $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$. *Find the general solution to this system of differential equations. What is the dominant eigenvector (and eigenvalue)?*

Differential equations and difference equations are not the same, and there are no terms like $(-6)^n$ or 4^n in this solution. The general solution is $\mathbf{x} = c_1e^{4t}\mathbf{b}_1 + c_2e^{-6t}\mathbf{b}_2$. (With our initial conditions, c_1 would be 4 and c_2 would be 1, but the question asked for a general solution.) Since $4 > -6$, the dominant eigenvector is $\mathbf{b}_1$ and the dominant eigenvalue is 4. Put another way, e^{4t} grows, while e^{-6t} shrinks, so the dominant term is the first one.

This problem was meant to illustrate why we have different criteria for finding the dominant mode for differential equations and for difference equations.

6. *(8 points) Let V be the subspace of $\mathbf{R}^4$ spanned by the three vectors* $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \\ -1 \end{pmatrix}$, $\mathbf{x}_3 = \begin{pmatrix} 5 \\ 2 \\ 1 \\ 0 \end{pmatrix}$. *Find an orthogonal basis for V.*

This is Gram-Schmidt. $\mathbf{y}_1 = \mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \end{pmatrix}$

$$\mathbf{y}_2 = \mathbf{x}_2 - \tfrac{\mathbf{x}_2 \cdot \mathbf{y}_1}{\mathbf{y}_1 \cdot \mathbf{y}_1} \mathbf{y}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \\ -1 \end{pmatrix} - \tfrac{6}{6} \begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

$$\mathbf{y}_3 = \mathbf{x}_3 - \tfrac{\mathbf{x}_3 \cdot \mathbf{y}_1}{\mathbf{y}_1 \cdot \mathbf{y}_1} \mathbf{y}_1 - \tfrac{\mathbf{x}_3 \cdot \mathbf{y}_2}{\mathbf{y}_2 \cdot \mathbf{y}_2} \mathbf{y}_2 = \begin{pmatrix} 5 \\ 2 \\ 1 \\ 0 \end{pmatrix} - \tfrac{6}{6} \begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \end{pmatrix} - \tfrac{8}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \\ 0 \end{pmatrix}.$$

Our orthogonal basis is $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$.

The most common mistake was using $\mathbf{x}_1$ and $\mathbf{x}_2$ instead of $\mathbf{y}_2$ and $\mathbf{y}_1$ in the computation of $\mathbf{y}_3$.

(10 points) 7a. Find all least-square solutions to $\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 7 \\ 5 \\ 7 \\ 13 \end{pmatrix}$.

$A^T A = \begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix}$ and $A^T \mathbf{b} = \begin{pmatrix} 32 \\ 90 \end{pmatrix}$. We get our least-squares solution by row reducing $\begin{pmatrix} 4 & 10 & | & 32 \\ 10 & 30 & | & 90 \end{pmatrix}$ to $\begin{pmatrix} 1 & 0 & | & 3 \\ 0 & 1 & | & 2 \end{pmatrix}$, hence $\hat{\mathbf{x}} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Most people got this right, although there were a lot of arithmetic errors, either in computing $A^T A$ or $A^T \mathbf{b}$ or in computing $(A^T A)^{-1}$.

b. Let W be the plane in $\mathbf{R}^4$ spanned by $\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ *and* $\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$. *Find the point in W closest to* $\begin{pmatrix}7\\5\\7\\13\end{pmatrix}$.

The closest point in $Col(A)$ to $\mathbf{b}$ is $\hat{\mathbf{b}} = A\hat{\mathbf{x}} = \begin{pmatrix}5\\7\\9\\11\end{pmatrix}$. You can also get this answer by finding an orthogonal basis for W (by Gram-Schmidt) and then using our formulas for projection.

The majority of the class got this one wrong. Note that $\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$ are *not* orthogonal, so you can't just project onto these two vectors and add! Also, note that this problem, while closely related to part (a), is not the same. The answer to (a) is $\hat{\mathbf{x}}$, while the answer to (b) is $A\hat{\mathbf{x}}$.

8. True/False (20 points, 2 pages):

a. If the columns of a square matrix are linearly independent, then the matrix is invertible.

True. The matrix has rank n.

b. If the columns of a square matrix are linearly dependent, then 0 is an eigenvalue of that matrix.

True. There is a nontrivial solution to $A\mathbf{x} = 0 = 0\mathbf{x}$.

c. The product $A\mathbf{x}$ of a matrix A with a vector $\mathbf{x}$ is a linear combination of the columns of A.

True. That's how we defined the product of a matrix and a vector.

d. Let A and B be matrices such that the product AB makes sense. The null space of B is a subspace of the null space of AB.

True. If $B\mathbf{x} = 0$, then $AB\mathbf{x} = A(B\mathbf{x}) = 0$.

e. The rank of a matrix is the number of linearly independent rows it has.

True. It's also the number of linearly independent columns. They're the same, since the column space and the row space have the same dimension.

f. If A is a 3×4 matrix and $\mathbf{b} \in \mathbf{R}^3$, then $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

False. Although there is a free variable, there might not be any solutions.

g. If W is a subspace of $\mathbf{R}^n$ and $\mathbf{x} \in \mathbf{R}^n$, then there is exactly one way to write $\mathbf{x}$ as the sum of a vector in W and a vector in $W^\perp$.

True. This decomposition is the point of doing projections.

h. For problems of the form $\mathbf{x}(n+1) = A\mathbf{x}(n)$, the dominant eigenvalue of A is the eigenvalue with greatest real part.

False. It's the eigenvalue with the greatest norm. The greatest real part applies to systems $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ of differential equations.

i. The geometric multiplicity of an eigenvalue is at least one and is at most the algebraic multiplicity.

True.

j. The system of equations $A\mathbf{x} = \mathbf{b}$ always has a least-squares solution.

True, since there is always a point in the column space of A that is closest to $\mathbf{b}$.