M340L Final Exam, May 13, 2010

1. (10 points) The matrix $A = \begin{pmatrix} 1 & 1 & 3 & 1 & 7 & 7 \\ 1 & 2 & 5 & 3 & 20 & 16 \\ 2 & 4 & 10 & 7 & 45 & 36 \end{pmatrix}$ is row-equivalent to $\begin{pmatrix} 1 & 0 & 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 5 & 4 \end{pmatrix}$.

a) Are the vectors $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$, and $\begin{pmatrix} 3 \\ 5 \\ 10 \end{pmatrix}$ linearly independent?

b) Do the vectors $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 5 \\ 10 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}$, $\begin{pmatrix} 7 \\ 20 \\ 45 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ 16 \\ 36 \end{pmatrix}$ span $\mathbf{R}^3$?

c) Find bases for the column space of A, for the row space of A, and for the null space of A.

2 (15 points) Let $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ be a basis for $\mathbf{R}^3$, and let $\mathcal{E}$ be the standard basis.

a. Compute the change-of-basis matrices $P_{\mathcal{E}\mathcal{B}}$ and $P_{\mathcal{B}\mathcal{E}}$.

b. If $\mathbf{x} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$, what is $[\mathbf{x}]_{\mathcal{B}}$?

c. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be a linear transformation given by the formula $T(\mathbf{x}) = \begin{pmatrix} 3x_1 + 2x_2 + x_3 \\ 2x_1 - x_3 \\ x_2 \end{pmatrix}$. Find the standard matrix of T (relative to the standard basis).

d. Find the matrix of T relative to the $\mathcal{B}$ basis.

3. (12 points)

For each of these square matrices, either find the inverse or explain why the inverse does not exist.

(a) $\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 3 & 5 \\ 2 & 1 & 4 \\ 3 & 4 & 9 \end{pmatrix}$

c) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & 4 & 5 \end{pmatrix}$

4. (10 points) (a) Write down the characteristic equation of the matrix $A = \begin{pmatrix} 3 & 7 \\ 1 & 4 \end{pmatrix}$. You do not need to find the eigenvalues or eigenvectors.

b) The eigenvalues of $B = \begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix}$ are 3 and -3. Find bases for E_3 and E_{-3}. Is B diagonalizable?

5. (15 points) The matrix $\begin{pmatrix} 3 & 3 \\ 3 & -5 \end{pmatrix}$ has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = -6$ and corresponding eigenvectors $\mathbf{b}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{b}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

a. Find the coordinates of $\begin{pmatrix} 13 \\ 1 \end{pmatrix}$ in the $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ basis.

b. If $\mathbf{x}(n+1) = A\mathbf{x}(n)$ and $\mathbf{x}(0) = \begin{pmatrix} 13 \\ 1 \end{pmatrix}$, find $\mathbf{x}(n)$ for all n. What is the dominant eigenvector (and eigenvalue) for this problem?

c. Suppose instead that $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$. Find the general solution to this system of differential equations. What is the dominant eigenvector (and eigenvalue)?

6. (8 points) Let V be the subspace of $\mathbf{R}^4$ spanned by the three vectors $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \\ -1 \end{pmatrix}$, $\mathbf{x}_3 = \begin{pmatrix} 5 \\ 2 \\ 1 \\ 0 \end{pmatrix}$. Find an orthogonal basis for V.

(10 points) 7a. Find all least-square solutions to $\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 7 \\ 5 \\ 7 \\ 13 \end{pmatrix}$.

b. Let W be the plane in $\mathbf{R}^4$ spanned by $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$. Find the point in W closest to $\begin{pmatrix} 7 \\ 5 \\ 7 \\ 13 \end{pmatrix}$.

8. True/False (20 points, 2 pages):

a. If the columns of a square matrix are linearly independent, then the matrix is invertible.

b. If the columns of a square matrix are linearly dependent, then 0 is an eigenvalue of that matrix.

c. The product $A\mathbf{x}$ of a matrix A with a vector $\mathbf{x}$ is a linear combination of the columns of A.

d. Let A and B be matrices such that the product AB makes sense. The null space of B is a subspace of the null space of AB.

e. The rank of a matrix is the number of linearly independent rows it has.

f. If A is a 3×4 matrix and $\mathbf{b} \in \mathbf{R}^3$, then $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

g. If W is a subspace of $\mathbf{R}^n$ and $\mathbf{x} \in \mathbf{R}^n$, then there is exactly one way to write $\mathbf{x}$ as the sum of a vector in W and a vector in $W^\perp$.

h. For problems of the form $\mathbf{x}(n+1) = A\mathbf{x}(n)$, the dominant eigenvalue of A is the eigenvalue with greatest real part.

i. The geometric multiplicity of an eigenvalue is at least one and is at most the algebraic multiplicity.

j. The system of equations $A\mathbf{x} = \mathbf{b}$ always has a least-squares solution.