Midterm distributions (all scores out of 115)

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Histogram[orscores["Midterm 1"], {0, 115, 5}]
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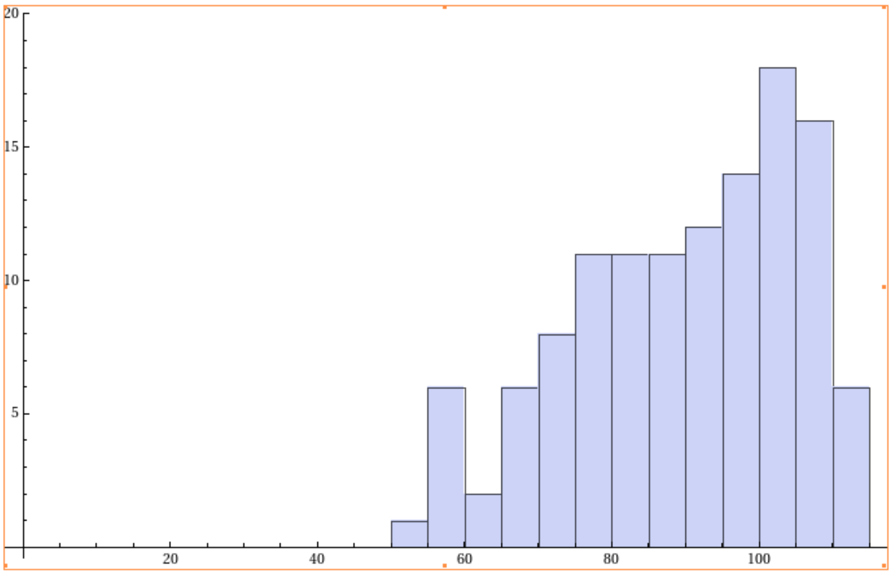


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Mean[orscores["Midterm 1"]] // N
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89.7742

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Median[orscores["Midterm 1"]]
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92

Lecture 11

Notes from Midterm 1:

1 a) $A = \begin{bmatrix} -3 & -4 & 6 \\ 1 & 2 & 0 \\ 0 & 2 & 6 \end{bmatrix}$ $\quad \vec{b} = \begin{bmatrix} -1 \\ 3 \\ 8 \end{bmatrix}$

Find sol^n's of $A\vec{x} = \vec{b}$ (in parametric form).

Popular error:

$$\left[\begin{array}{ccc|c} -3 & -4 & 6 & -1 \\ 1 & 2 & 0 & 3 \\ 0 & 2 & 6 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x_1 + 2x_2 = 3 \\ x_2 + 3x_3 = 4 \\ x_3 \text{ free} \end{array}$$

$$\begin{array}{l} x_1 = 3 - 2x_2 \\ x_2 = 4 - 3x_3 \\ x_3 \text{ free} \end{array}$$

$$\vec{x} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} x_3$$

This is not the parametric desc. of the sol^n set, b/c x_2 is not a free variable!

b) Find sol^n's of $A\vec{x} = \vec{b}$ $\quad \vec{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Row reduction $\rightsquigarrow$ inconsistent

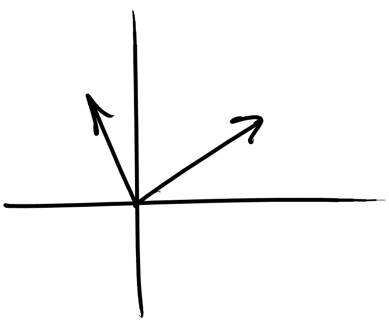
c) Do cols. of A span $\mathbb{R}^3$?

No: A has only 2 pivots, but 3 rows

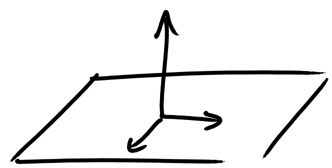
Name a vector in $\mathbb{R}^3$ not in the span of cols. of A:

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not in the span (because the eq. $A\vec{x} = \vec{b}$ is inconsistent as seen in previous part)

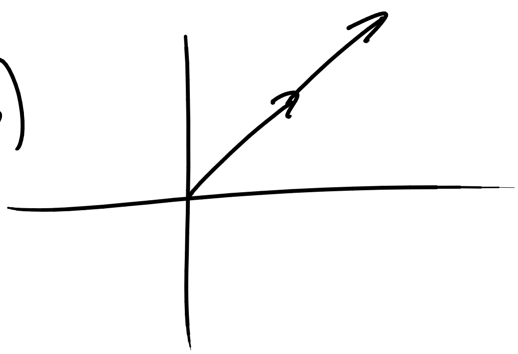
3a)



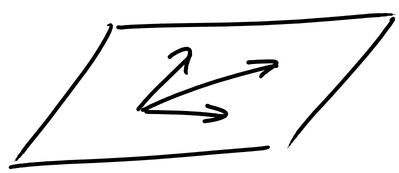
c)



b)



d)



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4) a) $A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \\ 0 & 0 \end{bmatrix}$ c) $B = \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}$

$T(\vec{x}) = A\vec{x}$ $U(\vec{x}) = B\vec{x}$

d) Find C s.t. $T(U(\vec{x})) = C\vec{x}$:

$$T(U(\vec{x})) = T(B\vec{x}) = A(B\vec{x}) = (AB)\vec{x}$$

i.e. $\underline{C = AB}$

e) $D = B^2$

6e), g)

e) A, B diagonal matrices B invertible

$$B^{-1}AB = A \ ?$$

This eq. is equivalent to
$$B(B^{-1}AB) = BA$$
$$AB = BA$$

This is true because A, B are diagonal matrices! T

g) A 3×5 matrix w/3 pivots

$$3\left\{\overbrace{\begin{bmatrix} \quad & \quad & \quad & \quad & \quad \\ & & & & \\ & & & & \end{bmatrix}}^{5}\right.$$

Does $T(\vec{x}) = A\vec{x}$ have image = the whole $\mathbb{R}^3$?

Yes, b/c it has pivot in every row T

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7) $T: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad \vec{u}, \vec{v}$ two vectors $\{\vec{u}, \vec{v}\}$ lin. indep.

$$T(\vec{u}) = \vec{v}$$
$$T(\vec{v}) = \vec{u}$$

a) Find some nonzero $\vec{w}$ s.th $T(\vec{w}) = \vec{w}$.

$$\boxed{\vec{w} = \vec{u} + \vec{v}}$$

has $T(\vec{w}) = T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) = \vec{v} + \vec{u} = \vec{w}$

ie $T(\vec{w}) = \vec{w}$

A lot of ppl wrote $\vec{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

This is right <u>if</u> we suppose $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

b) Find A s.t. $T(T(\vec{x})) = A\vec{x}$.

We know $T(T(\vec{u})) = \vec{u}$

$T(T(\vec{v})) = \vec{v}$

But <u>any</u> vector $\vec{y}$ in $\mathbb{R}^2$ can be written as $\vec{y} = x_1\vec{u} + x_2\vec{v}$.

$$T(T(\vec{y})) = x_1 T(T(\vec{u})) + x_2 T(T(\vec{v}))$$
$$= x_1\vec{u} + x_2\vec{v}$$
$$= \vec{y}$$

So $T(T(\vec{y})) = I\vec{y}$

i.e. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Last time:

Determinants: for a square matrix A we defined $\det A$.

Several ways of calculating:

- Cofactor expansion down any row or column

Ex Find $\det \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 3 & 2 \end{bmatrix}}_{A}$.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Pick the middle row:

$$\det A = -1 \cdot \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 0$$

$$= -1 \cdot (2 \cdot 2 - 1 \cdot 3) + 1 \cdot (1 \cdot 2 - 1 \cdot 0)$$

$$= -1 \cdot (1) + 1 \cdot (2)$$

$$= \underline{\underline{1}}$$

- Row reduction: if B is obtained from A by
  - adding a multiple of one row to another: $\det A = \det B$
  - exchanging two rows: $\det A = -\det B$
  - multiplying a row by k: $\det B = k \cdot \det A$

So row-reduce A to a B which is upper triangular: then $\det B$ is just the product of the diagonal entries and use that to get $\det A$

Ex $A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ -3 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 11 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix} = B$

$$\det A = \det B = 1 \cdot 11 \cdot 4 \cdot 7$$

Ex $A = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 3 & 0 \\ 2 & 0 & 0 \end{bmatrix} \sim B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

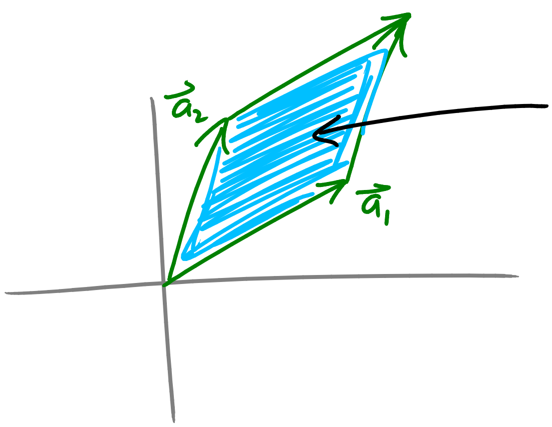
$$\det A = -\det B = -4 \cdot 3 \cdot 2 = \underline{-24}$$

- $\det(AB) = \det(A)\det(B)$
- $\det(A) \neq 0 \iff A$ is <u>invertible</u>

What does $\det(A)$ mean?

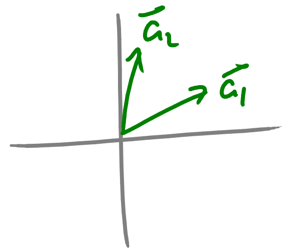
Start with 2×2 matrices: $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix}$ $\quad \vec{a}_1 \in \mathbb{R}^2$, $\vec{a}_2 \in \mathbb{R}^2$

Fact: $|\det A| =$ the area of the parallelogram with two sides $\vec{a}_1, \vec{a}_2$.

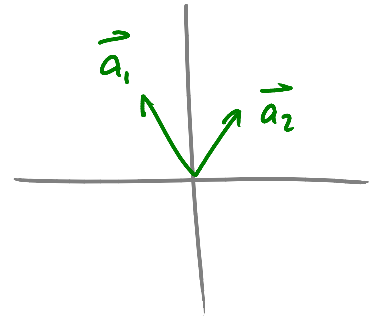


area $= |\det A|$

(Fact:
- $\det A > 0$ if $\vec{a}_2$ is reached by a <u>counterclockwise</u> rotation of $\vec{a}_1$ by $< 180°$
- $\det A < 0$ if $\vec{a}_2$ is reached by a <u>clockwise</u> rotation of $\vec{a}_1$ by $< 180°$
)



det $A > 0$

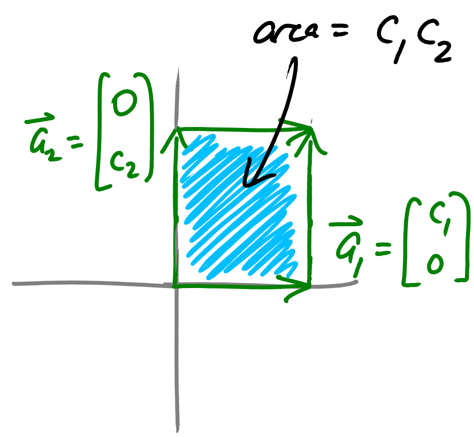


det $A < 0$

Why? Begin with $A = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$.

Then det $A = c_1 c_2$

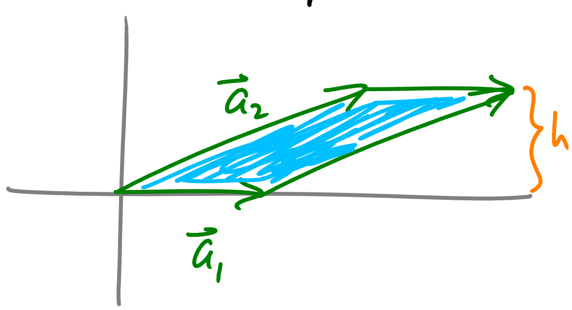
and the parallelogram is easy to study:



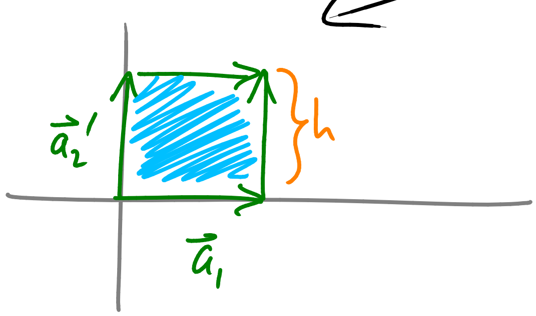
So in this case indeed

$$\det A = \text{area}(\text{parallelogram w/sides } \vec{a}_1, \vec{a}_2).$$

More generally say $\vec{a}_1, \vec{a}_2$ as in the figure:



has the <u>same area</u> as the rectangle



But the rectangle has area

$$= \det \begin{bmatrix} \vec{a}_1 & \vec{a}_2{}' \end{bmatrix}$$

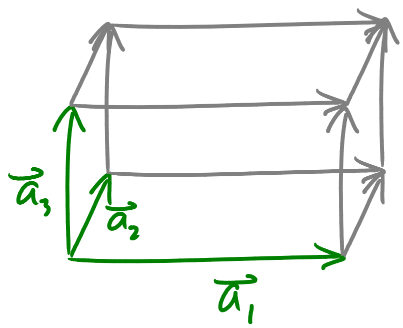
And $\det \begin{bmatrix} \vec{a}_1 & \vec{a}_2{}' \end{bmatrix} = \det \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix}$

because $\vec{a}_2' = \vec{a}_2 + c\vec{a}_1$ for some c

A similar statement for $\mathbb{R}^3$:

Suppose $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix}$ for $\vec{a}_1, \vec{a}_2, \vec{a}_3 \in \mathbb{R}^3$

Then $|\det(A)| =$ volume of a parallelepiped 3 of whose edges are $\vec{a}_1, \vec{a}_2, \vec{a}_3$.



volume of this parallelepiped $= |\det(A)|$