Lecture 18

28 Oct 2010

Midterm 2 1 week from today
Practice probs + old exam now posted
HW9 due Tuesday (short) will be posted shortly

Last time: eigenvalues + eigenvectors
$n\times n$ matrix A: eigenvector of A is a vector $\vec{v}$ w/ $A\vec{v}=\lambda\vec{v}$
for some scalar λ. λ is called the eigenvalue of A corresp. to $\vec{v}$.
[Often rewrite the eq. as $(A-\lambda I)\vec{v}=\vec{0}$].

In this lecture we're interested in producing a basis for $\mathbb{R}^n$ consisting of eigenvectors of A.

Fact: If $\{\vec{v}_1,\dots,\vec{v}_p\}$ are eigenvectors of A
with corresp. eigenvalues $\lambda_1,\dots,\lambda_p$
and all λ_i are distinct ($\lambda_i\neq\lambda_j$ when $i\neq j$)
Then $\{\vec{v}_1,\dots,\vec{v}_p\}$ are linearly independent.

Say A,B are $n\times n$ matrices and $A=PBP^{-1}$ for some (invertible) matrix P.
Then we say A and B are similar.

Fact: If A and B are similar, then A and B have the same eigenvalues.

[Why? The eigenvalues of A are solutions of characteristic eq $\det(A-\lambda I)=0$
" " " B " " " " " $\det(B-\lambda I)=0$

But these 2 eq. are the same:
because $\det(A-\lambda I)=\det(PBP^{-1}-\lambda I)$
$=\det(PBP^{-1}-\lambda PIP^{-1})$

$$= \det\left[P(B-\lambda I)P^{-1}\right]$$
$$= (\det P)\det(B-\lambda I)(\det P^{-1})$$
$$= \det(B-\lambda I)$$

Now suppose A is similar to a <u>diagonal</u> matrix D:

$$A = PDP^{-1}$$

Then $A^2 = (PDP^{-1})^2 = (PDP^{-1})(PDP^{-1})$
$$= PDIDP^{-1}$$
$$= PD^2P^{-1}$$

similarly $A^n = PD^nP^{-1}$

And the right side is <u>easy</u> to calculate!

Ex $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. $A = PDP^{-1}$ when $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$

(use formula for inverse of 2×2 matrix)

Then $A^n = PD^nP^{-1}$
$$= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^n & 0 \\ 0 & 3^n \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 2\cdot 5^n - 3^n & 5^n - 3^n \\ 2\cdot 3^n - 2\cdot 5^n & 2\cdot 3^n - 5^n \end{bmatrix}$$

When can we do this?

Say A is <u>diagonalizable</u> if A is similar to a diagonal matrix.

<u>Fact</u> $n\times n$ matrix A is diagonalizable <u>if and only if</u> A has n lin. indep. eigenvectors.

(i.e. there's a basis of $\mathbb{R}^n$ consisting of eigenvectors of A)

Fact $A = PDP^{-1}$ is equivalent to saying the columns of P are n linearly independent eigenvectors of A.
(And the entries of D are the corresponding eigenvalues.)

Writing a matrix A in the form $A = PDP^{-1}$ for D diagonal is called diagonalizing A.

Ex Diagonalize the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$.

Step 1: Find the eigenvalues of A.

Char eq: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = 0$$

Calculating this determinant we get (e.g. by cofactors)

$$\det(A - \lambda I) = -\lambda^3 - 3\lambda^2 + 4$$

Luckily can see by eye $\lambda = 1$ is a root. Then can factor:

$$\det(A - \lambda I) = -(\lambda - 1)(\lambda^2 + 4\lambda + 4)$$
$$= -(\lambda - 1)(\lambda + 2)^2$$

So the eigenvalues of A are $\lambda = 1$ and $\lambda = -2$.

Step 2 Find 3 linearly independent eigenvectors of A.

Eigenvectors with $\lambda = 1$: $A\vec{x} = 1\vec{x}$

$$(A - 1 \cdot I)\vec{x} = \vec{0}$$

$$\begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix} \vec{x} = \vec{0}$$

Solving this for $\vec{x}$ we get $\vec{x} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Pick one eigenvector: say $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$$\left(\text{Check: } A\vec{v} = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \vec{v} \right)$$

Eigenvectors with $\lambda = -2$:

$$(A + 2I)\vec{x} = \vec{0}$$

$$\begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \vec{x} = \vec{0}$$

Solve: reduce to $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0}$ $\quad x_1 + x_2 + x_3 = 0$

x_2 free

x_3 free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So this eigenspace is 2-dimensional.

We want to find 2 lin. indep. vectors in there:

take $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $\quad \vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

So altogether we've found 3 lin. indep. eigenvectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

w/ eigenvalues

$$\lambda_1 = 1 \quad \lambda_2 = -2 \quad \lambda_3 = -2$$

Step 3 $P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Step 4 $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Ex Diagonalize $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ if possible.

Step 1 Find eigenvalues: $\lambda = 3, 4$ (b/c A is (upper) triangular so its e-vals are its diag. entries)

Step 2 Find eigenvectors:

$\underline{\lambda=4}$: $A - 4I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Solve $(A-4I)\vec{x} = \vec{0}$: get $\vec{x} = t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

So e.g. $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvect. w/ eigenval. 4.

$\underline{\lambda=3}$: $A - 3I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solve $(A-3I)\vec{x} = \vec{0}$: $A - 3I \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Rank 2, so space of solutions $[\mathrm{Nul}(A-3I)]$ is only 1-dimensional

So we will not be able to find 2 lin. indep. eigenvect. w/ $\lambda = 3$!

So A is not diagonalizable.

Aside: from PoV of diagonalizing, the eigenvalue $\lambda=0$ is just as good as any other eigenvalue.

[But NB: A has $\lambda=0$ for an eigenval
if and only if
A is not invertible]

Ex Diagonalize $A=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$ if possible.

Step 1 $\det(A-\lambda I) = \begin{vmatrix}-\lambda & 1\\ -1 & -\lambda\end{vmatrix} = \lambda^2+1$

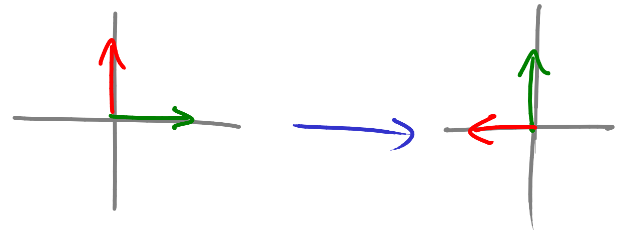
$\lambda^2+1=0$ $\qquad$ $\lambda=\pm i$

There are no (real) eigenvectors with non-real eigenvalues!

So: A is not diagonalizable.

[Why doesn't A have any eigenvectors?
A is the std matrix for a rotation by $90°$

So A always changes the direction!


]

Fact If A is $n \times n$ and has n distinct eigenvalues then A is diagonalizable.

If A has $< n$ eigenvalues then A still might be diagonalizable.

Fact Say A has eigenvalues $\lambda_1, \ldots, \lambda_p$

- The dimension of each eigenspace is $\leq$ the multiplicity of the corresponding eigenvalue as a root of the char. eq.

- A is diagonalizable if and only if the dimension of each eigenspace is equal to the multiplicity of the corresp. eigenvalue as a root of the char. eq.