Lecture 21

11 Nov 2010

NB: The grader erred in marking Sec 5.3 #11 — you should have gotten full credit no matter which order you put the eigenvalues in! (+2)

Perspectives (aka: things I should have said in earlier lectures, but didn't)

- Another way to think about change-of-basis: Suppose $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ and $\mathcal{C} = \{\vec{c}_1, \dots, \vec{c}_n\}$ are two bases for $\mathbb{R}^n$. Then $P_\mathcal{B} = [\vec{b}_1 \cdots \vec{b}_n]$ $P_\mathcal{C} = [\vec{c}_1 \cdots \vec{c}_n]$

$$\left.\begin{array}{l} \vec{x} = P_\mathcal{B}[\vec{x}]_\mathcal{B} \\ \vec{x} = P_\mathcal{C}[\vec{x}]_\mathcal{C} \end{array}\right\} \Rightarrow P_\mathcal{B}[\vec{x}]_\mathcal{B} = P_\mathcal{C}[\vec{x}]_\mathcal{C} \quad \text{so} \quad [\vec{x}]_\mathcal{B} = P_\mathcal{B}^{-1} P_\mathcal{C} [\vec{x}]_\mathcal{C}$$

Now, we said before that $[\vec{x}]_\mathcal{B} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P} [\vec{x}]_\mathcal{C}$

So: $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = P_\mathcal{B}^{-1} P_\mathcal{C}$

(This is also an instance of $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \underset{\mathcal{B}\leftarrow\mathcal{E}}{P}\ \underset{\mathcal{E}\leftarrow\mathcal{C}}{P}$)

Another thing from previous lectures:

If D is a diagonal matrix then its eigenvalues (w/multiplicity) are just its diagonal entries, and $\det D$ is just the <u>product</u> of the diagonal entries.

So for diag. matrices, <u>the determinant is the product of the eigenvalues (w/mult).</u>

In fact, this is true even for <u>diagonalizable</u> matrices:

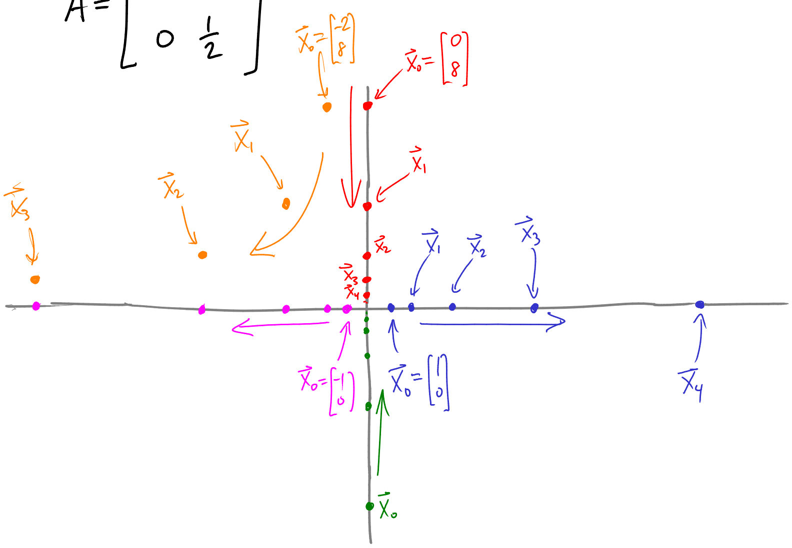
If $A = PDP^{-1}$ then eigenvalues (A) = eigenvalues (D)

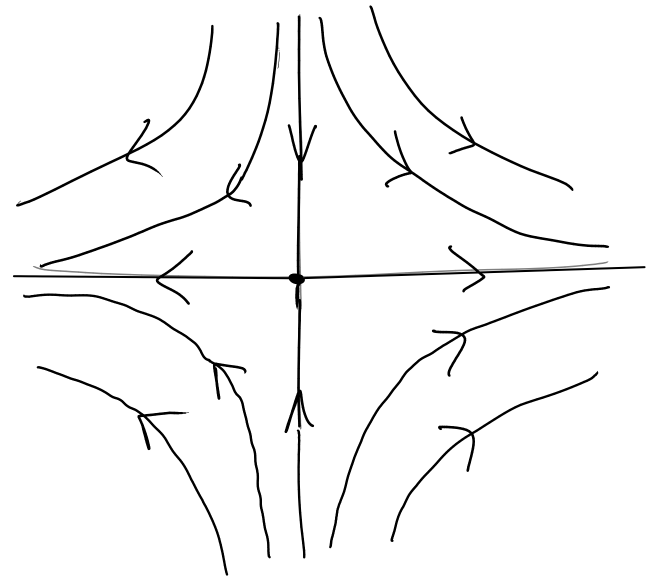
so product of eigenvalues (A) = product of eigenvalues (D)

$= \det(D) = \det(A)$!

Last time:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \qquad \vec{x}_n = A\vec{x}_{n-1} = A^n \vec{x}_0$$



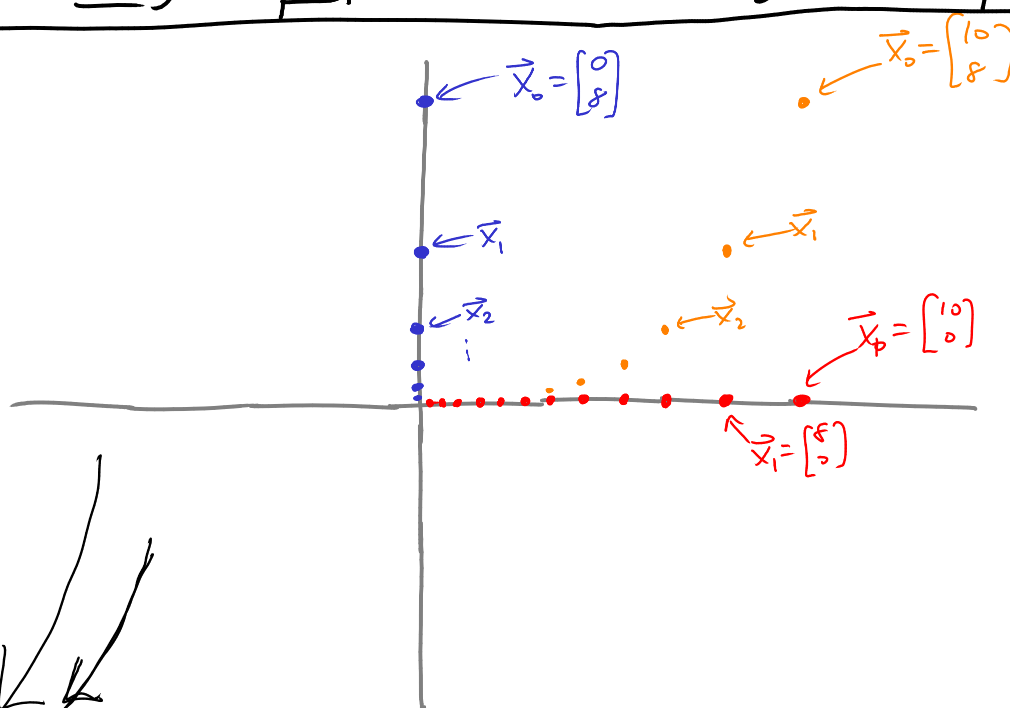


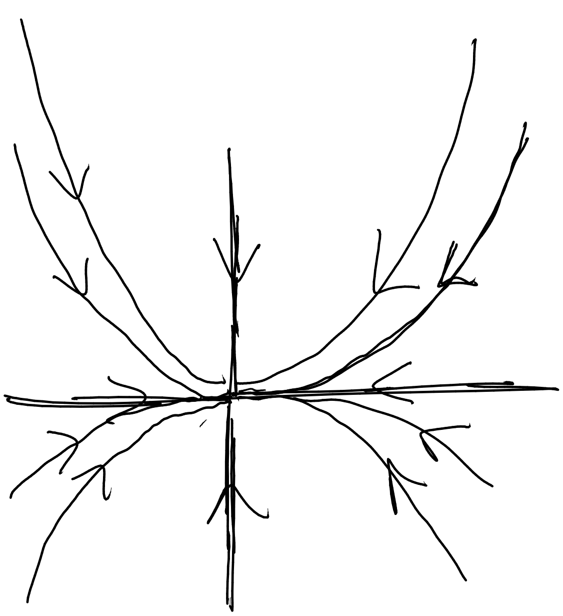
This is a "saddle point": one <u>attracting</u> direction
one <u>repelling</u> direction

(In gen'l, whenever we have both <u>attracting</u> and <u>repelling</u> directions we call the origin a <u>saddle pt</u>)

Another example:

$$A = \begin{bmatrix} 4/5 & 0 \\ 0 & 1/2 \end{bmatrix}$$

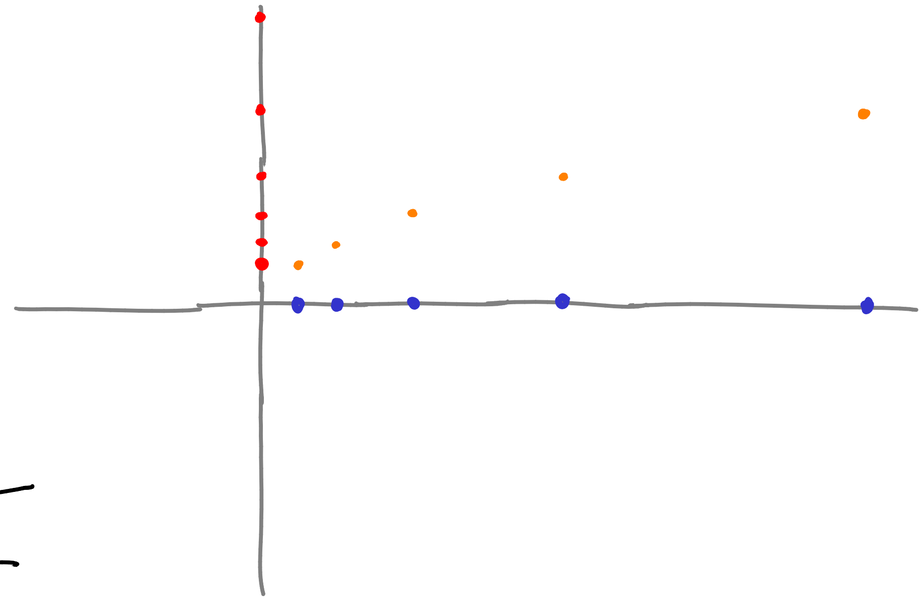


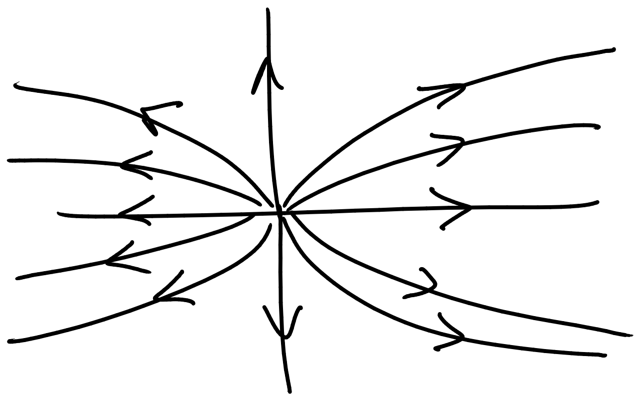


<u>Attracting fixed point.</u>

the direction of <u>fastest</u> attraction is the y-axis, corresponding to the <u>smallest</u> eigenvalue $(\lambda = \frac{1}{2})$.

Ex $A = \begin{bmatrix} 2 & 0 \\ 0 & 3/2 \end{bmatrix}$





Repelling fixed point; direction of fastest repulsion is the x-axis, corresp. to the biggest eigenvalue ($\lambda = 2$).

Now suppose we have the dynamical sys. $\vec{x}_{k+1} = A\vec{x}_k$ where A is not diagonal but A is diagonalizable ($n \times n$ matrix).

Basis of eigenvectors for A: $\{\vec{v}_1, \dots, \vec{v}_n\}$

Define $P = \begin{bmatrix} \vec{v}_1 \cdots \vec{v}_n \end{bmatrix}$ and $\vec{y}_k = P^{-1}\vec{x}_k$

Since $A = PDP^{-1}$ and $D = P^{-1}AP$:

$$\vec{x}_{k+1} = A\vec{x}_k$$
$$P^{-1}\vec{x}_{k+1} = P^{-1}A\vec{x}_k$$
$$\vec{y}_{k+1} = (P^{-1}AP)(P^{-1}\vec{x}_k)$$
$$\vec{y}_{k+1} = D\vec{y}_k$$

So any question about the dynamical sys. $\vec{x}_{k+1} = A\vec{x}_k$ can be answered by looking at the matrices P and D

Ex $\quad A\vec{x}_k = \vec{x}_{k+1} \qquad A = \begin{bmatrix} 5/4 & -3/4 \\ -3/4 & 5/4 \end{bmatrix}$

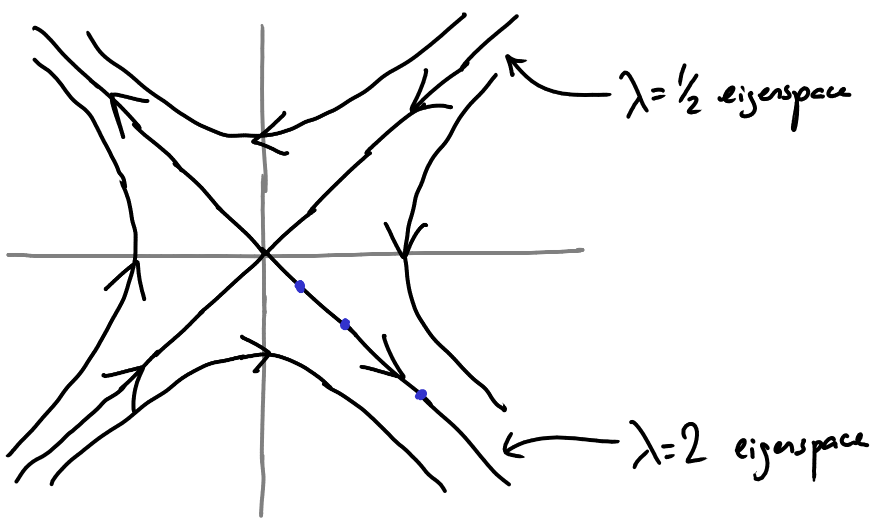
- Is the origin attracting, saddle or repelling?
- Plot the phase portrait.

Find a basis of eigenvectors for A: two eigenvalues $\lambda_1 = 2 \quad \lambda_2 = \frac{1}{2}$

$\lambda_1 = 2$: eigenspace $\text{Span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ — repelling direction

$\lambda_2 = \frac{1}{2}$: eigenspace $\text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ — attracting direction

One eigenvalue >1, one eigenvalue <1 $\Rightarrow$ origin is saddle point.

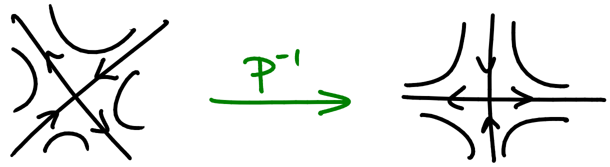


This is the picture in the $\vec{x}_k$ variables.

We could also change to the $\vec{y}_k$ variables: $\quad \vec{y}_{k+1} = D\vec{y}_k \quad D = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

$$\vec{y}_k = P^{-1}\vec{x}_k$$

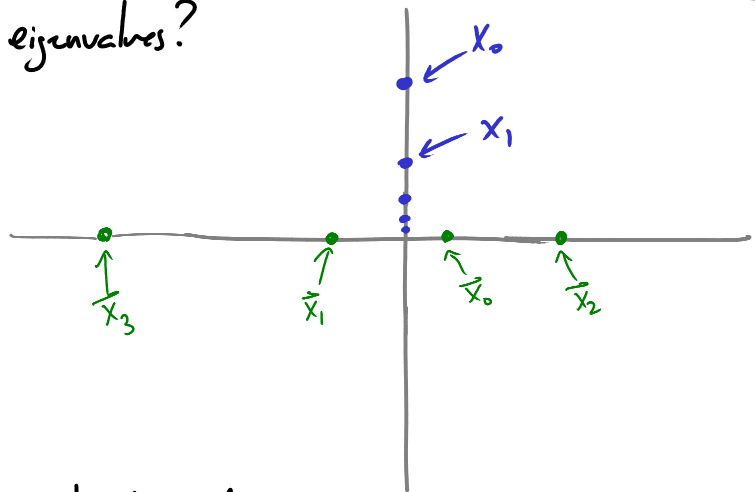
In those variables it looks like the sys we described at the beginning of lecture



$$\left[\left[P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \text{ which is } \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ with } \theta = -\frac{\pi}{4} \right]\right]$$

What about negative eigenvalues?

$$A = \begin{bmatrix} -2 & \\ & \frac{1}{2} \end{bmatrix}$$



Repelling dirs. have $|\lambda| > 1$

Attracting dirs. have $|\lambda| < 1$

Another possibility: A which is not diag'ble because its eigenvalues are complex

Ex $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\qquad \det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$

$$\lambda = \pm i$$

No real eigenvectors.

But, let's try using vectors with complex entries:

$\vec{v}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix} \qquad A\vec{v}_1 = \begin{bmatrix} +i \\ 1 \end{bmatrix} = i\vec{v}_1 \qquad \left(\begin{matrix} i = i \cdot 1 \\ 1 = i \cdot (-i) \end{matrix} \right)$

$\vec{v}_2 = \begin{bmatrix} 1 \\ +i \end{bmatrix} \qquad A\vec{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix} = -i\vec{v}_2$

So $\vec{v}_1$ is a (complex) eigenvector w/ eigenvalue $\lambda_1 = +i$

$\vec{v}_2$ " " " " " " $\lambda_2 = -i$

Ex Find the (complex) eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 1/2 & -3/5 \\ 3/4 & 11/10 \end{bmatrix}$$

$$\det(A - \lambda I) = 0 \quad \text{for} \quad \lambda = \frac{4}{5} \pm \frac{3}{5} i$$

So, 2 cplx eigenvalues $\lambda_1 = \frac{4}{5} + \frac{3}{5} i$

$\lambda_2 = \frac{4}{5} - \frac{3}{5} i$

To find eigenvectors:

for λ_2:

$$A\vec{x} = \lambda_1 \vec{x}$$

$$(A - \lambda_2 I)\vec{x} = 0$$

$$\begin{bmatrix} -3/10 + 3/5\, i & -3/5 \\ 3/4 & 3/10 + 3/5\, i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving by row reduction would require division of cplx #s $\Rightarrow$ annoying.

Alternative: write the 1st equation out - $\left(-\frac{3}{10} + \frac{3}{5} i\right) x_1 + \left(-\frac{3}{5}\right) x_2 = 0$

We already know the sys. is consistent and has 1-dim solution space $\Rightarrow$ if we solve this, we automatically solve the 2nd eq.

So solve:

$$(-3 + 6i)\, x_1 - 6 x_2 = 0$$

$$(-1 + 2i)\, x_1 = 2 x_2$$

$$\left(-\frac{1}{2}+i\right)x_1 = x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \left(-\frac{1}{2}+i\right)x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -\frac{1}{2}+i \end{bmatrix}$$