Lecture 22

16 Nov 2010

Exam 2 solutions now posted — check!

HW 10 due Thu as usual

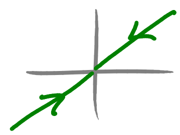
HW 11 due next Tue b/c of Thanksgiving (short)

Last 2 lectures will be given by Prof. Ray Heitmann (I am traveling :( )
(#25-26)

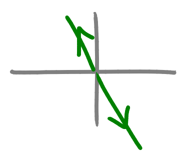
Last time: Dynamical systems $\vec{x}_{n+1} = A\vec{x}_n$

Fact: Eigenspaces of A with real λ and $|\lambda| \neq 1$ are attracting/repelling directions for the dynamical system:

- $|\lambda| < 1$: attracting



- $|\lambda| > 1$: repelling



Complex eigenvalues/eigenvectors

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{3}{5} \\ \frac{3}{4} & \frac{11}{10} \end{bmatrix} \qquad \det(A - \lambda I) = \lambda^2 - \frac{8}{5}\lambda + 1 = 0$$

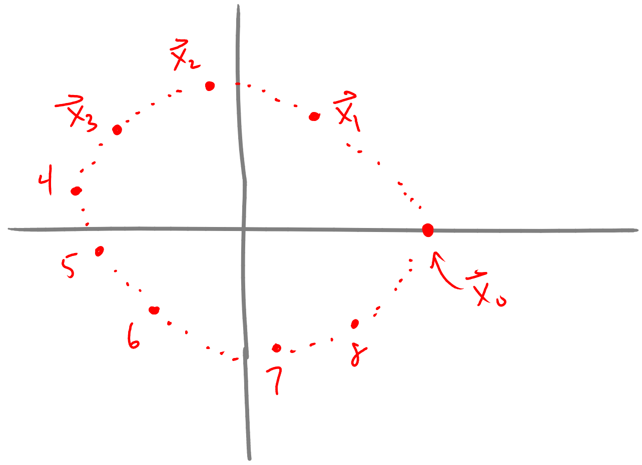
$$\Rightarrow \lambda = \frac{4}{5} \pm \frac{3}{5} i$$

$$A - \lambda I = \begin{bmatrix} \frac{1}{2} - \lambda & -\frac{3}{5} \\ \frac{3}{4} & \frac{11}{10} - \lambda \end{bmatrix}$$

aside: here we see that the cplx eigenvals come in a complex conjugate pair.
Even for bigger matrices, this always happens:
e.g. 100x100 matrix could have
80 real eigenvals
and 20 cplx ones, in 10 conjugate pairs

$$\lambda_1 = \frac{4}{5} + \frac{3}{5} i : \quad \vec{v}_1 = \begin{bmatrix} 1 \\ -\frac{1}{2} - i \end{bmatrix}$$

$$\lambda_2 = \frac{4}{5} - \frac{3}{5} i : \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -\frac{1}{2} + i \end{bmatrix}$$



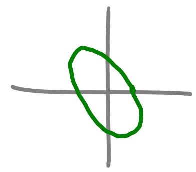
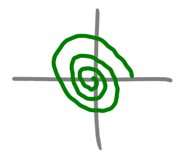
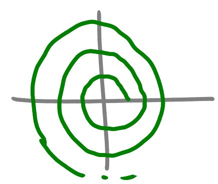
$$\vec{x}_{n+1} = A\vec{x}_n$$

$$\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{x}_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{4} \end{bmatrix}$$

$$\vdots$$

Fact: If A is a 2×2 matrix with complex eigenvalues $\lambda, \bar{\lambda}$, and $\vec{x}_{n+1} = A\vec{x}_n$:

- If $|\lambda| = 1$, the $\vec{x}_n$ all lie on an ellipse 
- If $|\lambda| < 1$, the $\vec{x}_n$ spiral into the origin 
- If $|\lambda| > 1$, the $\vec{x}_n$ spiral out of the origin 

(Recall that if λ is complex, $\lambda = a + bi$, then $|\lambda|$ means $\sqrt{a^2+b^2}$. So e.g. $\lambda = \frac{4}{5} + \frac{3}{5}i$ has $|\lambda| = \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2} = \frac{1}{5}\sqrt{25} = 1$)

Why?

Look at a model example:
$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\det(C - \lambda I) = \begin{vmatrix} a-\lambda & -b \\ b & a-\lambda \end{vmatrix}$$

$$= (a-\lambda)^2 + b^2 = 0$$

Could solve this by quadratic formula: but also can use difference of squares

$$= \left[(a-\lambda) + ib\right]\left[(a-\lambda) - ib\right]$$

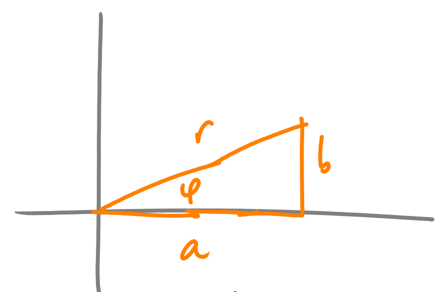
$$= \left[(a+ib) - \lambda\right]\left[(a-ib) - \lambda\right]$$

$$\lambda = a \pm ib$$

To understand how C acts on the plane:

Let $r = |\lambda| = \sqrt{a^2+b^2}$

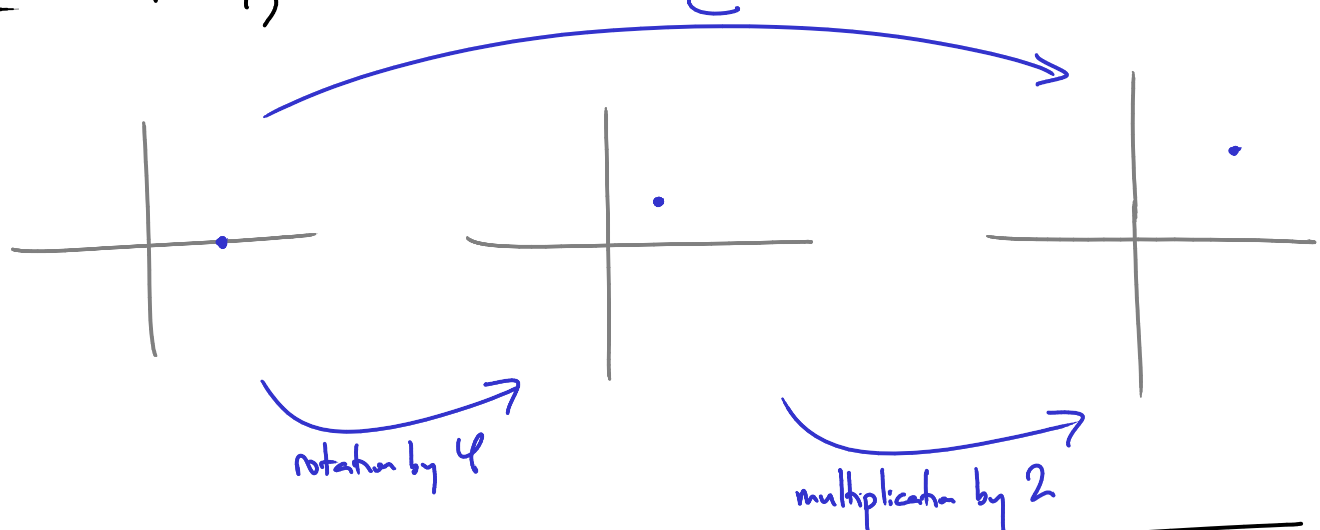
$\varphi = \tan^{-1}(b/a)$



$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r\begin{bmatrix} \frac{a}{r} & -\frac{b}{r} \\ \frac{b}{r} & \frac{a}{r} \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} \frac{a}{r} & -\frac{b}{r} \\ \frac{b}{r} & \frac{a}{r} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}}_{\text{scaling by } r}\underbrace{\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}}_{\text{rotation by } \varphi}$$

Ex if $\varphi = \pi/4$, $r=2$:



So if we define $\vec{y}_{n+1} = C\vec{y}_n$ then $\vec{y}_0, \vec{y}_1, \vec{y}_2, \dots$ will lie on:

- A circle if $r=1$ (i.e. $|\lambda|=1$) — only rotation, no scaling
- An inspiring spiral if $r<1$
- An outgoing spiral if $r>1$

That was special to the matrix C; but in fact any 2×2 matrix w/ complex eigenvalues is similar to C:

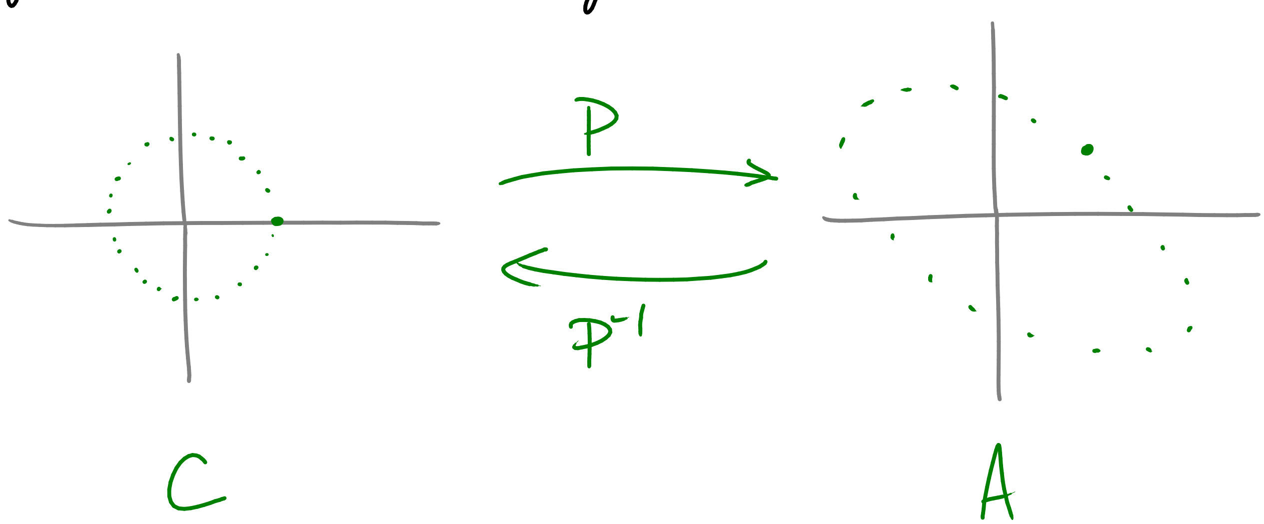
Fact: Say A is a 2×2 matrix with complex eigenvalues $\lambda = a-bi$, $\overline{\lambda} = a+bi$

and corresponding complex eigenvector $\vec{v}$ with $A\vec{v}=\lambda\vec{v}$

Then $A = PCP^{-1}$

where $P = \begin{bmatrix} \mathrm{Re}\,\vec{v} & \mathrm{Im}\,\vec{v} \end{bmatrix}$ $\qquad C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

(An analogue of diagonalization: pass from the "hard" system $\vec{x}_{n+1} = A\vec{x}_n$ to the "easier" one $\vec{y}_{n+1} = C\vec{y}_n$ by the change of variables $\vec{y} = P^{-1}\vec{x}$)



One more word about eigenvectors:

Say we begin w/ the system $\vec{X}_{n+1} = A\vec{X}_n$ $\qquad \vec{X}_n = \begin{bmatrix} x_n^{(1)} \\ x_n^{(2)} \\ x_n^{(3)} \end{bmatrix}$

e.g. if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

then
$$x_{n+1}^{(1)} = x_n^{(1)} + 2x_n^{(2)} + 3x_n^{(3)}$$
$$x_{n+1}^{(2)} = 4x_n^{(1)} + 5x_n^{(2)} + 6x_n^{(3)}$$
$$x_{n+1}^{(3)} = 7x_n^{(1)} + 8x_n^{(2)} + 9x_n^{(3)}$$

This looks complicated: all the variables are <u>mixed together</u> (coupled)!

But: if we <u>diagonalize</u> A, $A = PDP^{-1}$

and write $\vec{y}_n = P^{-1}\vec{x}_n$ (new set of variables) $\qquad \vec{y}_n = \begin{bmatrix} y_n^{(1)} \\ y_n^{(2)} \\ y_n^{(3)} \end{bmatrix}$

Then the sys. becomes $\vec{y}_{n+1} = D\vec{y}_n$

$$y_{n+1}^{(1)} = \lambda_1 y_n^{(1)}$$
$$y_{n+1}^{(2)} = \lambda_2 y_n^{(2)}$$
$$y_{n+1}^{(3)} = \lambda_3 y_n^{(3)}$$

Inner Products, Length and Orthogonality (Sec 6.1)

Say $\vec{u}, \vec{v}$ in $\mathbb{R}^n$.
$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Define the <u>dot-product</u>

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

<u>Ex</u> $\vec{u} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$ $\vec{u} \cdot \vec{v} = -1 + 8 - 10 = \underline{-3}$

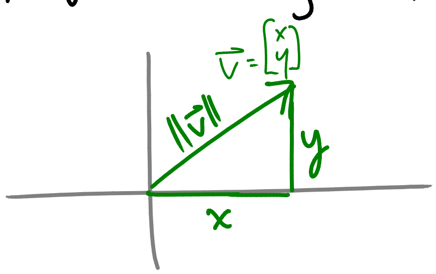
<u>Fact</u>
1) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
3) $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$
4) $\vec{u} \cdot \vec{u} \geq 0$, <u>and</u> $\vec{u} \cdot \vec{u} = 0$ if and only if $\vec{u} = \vec{0}$

[Why? 1), 2), 3) are "easy"

4) $\vec{u} \cdot \vec{u} = u_1^2 + u_2^2 + \cdots + u_n^2$ all $u_i^2 \geq 0$ so $\vec{u} \cdot \vec{u} \geq 0$
and if $\vec{u} \cdot \vec{u} = 0$ all $u_i = 0$
i.e. $\vec{u} = \vec{0}$]

We define the <u>length</u> of $\vec{v}$ by $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$

(e.g. if $\vec{v} \in \mathbb{R}^2$ this agrees w/ our usual notion of length:



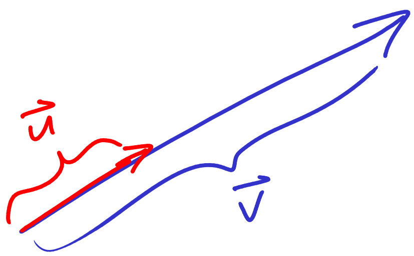
$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{x^2 + y^2}$$)

Fact $\|c\vec{v}\| = |c| \cdot \|\vec{v}\|$

[Why? $\|c\vec{v}\| = \sqrt{(c\vec{v})\cdot(c\vec{v})} = \sqrt{c^2(\vec{v}\cdot\vec{v})} = \sqrt{c^2}\cdot\sqrt{\vec{v}\cdot\vec{v}} = |c|\cdot\|\vec{v}\|$]

If $\|\vec{v}\| = 1$ we call $\vec{v}$ a <u>unit vector</u>.

For <u>any</u> $\vec{v} \neq \vec{0}$, we can make a unit vector $\vec{u} = \left(\dfrac{1}{\|\vec{v}\|}\right)\vec{v} = \dfrac{\vec{v}}{\|\vec{v}\|}$



$$\left(\|\vec{u}\| = \left\| \frac{\vec{v}}{\|\vec{v}\|} \right\| = \frac{\|\vec{v}\|}{\|\vec{v}\|} = 1 \right)$$

Ex Find a unit vector $\vec{u}$ in same direction as $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\sqrt{1^2+2^2+3^2}}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{bmatrix}$$

Distance in $\mathbb{R}^n$

Say $\vec{v}, \vec{w}$ in $\mathbb{R}^n$. Define their <u>distance</u> to be $\text{dist}(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\|$.

Ex $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}$ $\text{dist}(\vec{v},\vec{w}) = \left\| \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \right\| = \sqrt{17}$