Lecture 4

17 Sep 2010

Reminder: HW 2 due Thu (9 Sep)
My office hours Tue 9:30–10:30a (right after class)
(RLM 9.134) Fri 11:00a–12:00 noon

Last time: matrix equations $A\vec{x} = \vec{b}$
homogeneous lin sys $A\vec{x} = \vec{0}$ and their solution sets

Ex Say $A = \begin{bmatrix} 0 & 1 & -3 \\ 4 & 1 & 5 \\ 2 & 0 & 4 \end{bmatrix}$ Describe the sol'ns of $A\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccc|c} 0 & 1 & -3 & 0 \\ 4 & 1 & 5 & 0 \\ 2 & 0 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow \begin{array}{l} x_1 = -2x_3 \\ x_2 = 3x_3 \\ x_3 \text{ \underline{free}} \end{array}$$

i.e. $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ 3x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

i.e. $\vec{x} = t \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$ i.e. the sol'n set is $\text{Span}\left\{ \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \right\}$.

Fact: The sol'n set of a homog. lin. sys. (or homog. matrix eq.) is always of the form $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$ for some set of vectors $\vec{v}_1, \dots, \vec{v}_k$

(k = # of free variables)

Sol^{n}s of inhomogeneous systems

Ex Describe the sol^n set of $A\vec{x}=\vec{b}$

$$A=\begin{bmatrix}0 & 1 & -3\\ 4 & 1 & 5\\ 2 & 0 & 4\end{bmatrix} \qquad \vec{b}=\begin{bmatrix}-5\\ 15\\ 10\end{bmatrix}$$

Row reduce:
$$\left[\begin{array}{ccc|c}0 & 1 & -3 & -5\\ 4 & 1 & 5 & 15\\ 2 & 0 & 4 & 10\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 0 & 2 & 5\\ 0 & 1 & -3 & -5\\ 0 & 0 & 0 & 0\end{array}\right] \qquad \begin{aligned} x_1 &= -2x_3+5\\ x_2 &= 3x_3-5\\ x_3 &\text{ free}\end{aligned}$$

It's consistent

Sol^n set:
$$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}-2x_3+5\\ 3x_3-5\\ x_3\end{bmatrix}=x_3\begin{bmatrix}-2\\ 3\\ 1\end{bmatrix}+\begin{bmatrix}5\\ -5\\ 0\end{bmatrix}$$

ie
$$\vec{x}=\underbrace{t\begin{bmatrix}-2\\ 3\\ 1\end{bmatrix}}_{\vec{v}_h}+\underbrace{\begin{bmatrix}5\\ -5\\ 0\end{bmatrix}}_{\vec{p}}$$

$\vec{v}_h$: any solution of the homogeneous eq $A\vec{x}=\vec{0}$

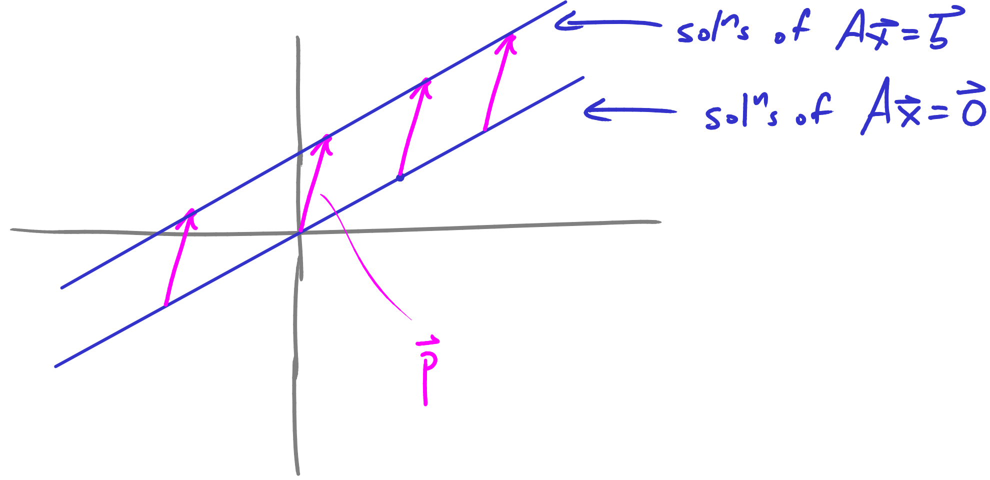
$\vec{p}$: one specific sol^n of the inhomog. eq $A\vec{x}=\vec{b}$

Fact: If the eq. $A\vec{x}=\vec{b}$ is consistent, then fix one solution $\vec{p}$. Then the solution set of $A\vec{x}=\vec{b}$ is the set of all vectors

$$\vec{x}=\vec{v}_h+\vec{p}$$

where $\vec{v}_h$ is any solution of the homog. eq. $A\vec{x}=\vec{0}$.

Ex



Linear Independence (Sec 1.7)

Call a set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ of vectors *linearly independent* if the equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}$$

has only the *trivial solution* (all $x_i = 0$).

Call $\{\vec{v}_1, \dots, \vec{v}_n\}$ *linearly dependent* if not lin. indep., i.e. if there exist $c_1, \dots, c_n$ not all zero such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}.$$

Ex $\left\{ \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 4 \\ -2 \end{bmatrix} \right\}$ is linearly dependent

because $1\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} - 2\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 7 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Ex $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is linearly independent

because $x_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

which can only be $\vec{0}$ if $x_1=0,\ x_2=0,\ x_3=0$

Ex Are $\left\{ \begin{bmatrix}4\\3\\-1\\-1\end{bmatrix}, \begin{bmatrix}14\\11\\-1\\-1\end{bmatrix}, \begin{bmatrix}2\\2\\2\\2\end{bmatrix} \right\}$ lin indep?

Look for (nontrivial) sol^n's of

$$x_1 \begin{bmatrix}4\\3\\-1\\-1\end{bmatrix} + x_2 \begin{bmatrix}14\\11\\-1\\-1\end{bmatrix} + x_3 \begin{bmatrix}2\\2\\2\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & 14 & 2 & 0 \\ 3 & 11 & 2 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 3x_3$
$x_2 = -x_3$
x_3 free

column w/o pivot $\Rightarrow$ free variable

Because there is a free var, this sys has nontriv solution
$\Rightarrow$ the vectors are linearly dependent.

What is the linear dependence relation? Pick say $x_3 = 1$:
then $x_1 = 3,\ x_2 = -1$

giving the relation $3\begin{bmatrix}4\\3\\-1\\-1\end{bmatrix} - \begin{bmatrix}14\\11\\-1\\-1\end{bmatrix} + \begin{bmatrix}2\\2\\2\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

Ex Are $\left\{ \begin{bmatrix}7\\3\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix} \right\}$ lin indep?

Look for nontriv sol of $x_1\begin{bmatrix}7\\3\end{bmatrix} + x_2\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$$\left[\begin{array}{cc|c} 7 & 1 & 0 \\ 3 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

No free vars $\Rightarrow$ no nontriv sol^{n} $\Rightarrow$ the vectors are lin. indep.

The eq. $x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{0}$

is equivalent to the matrix equation $A\vec{x} = \vec{0}$

with

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}$$

So: Fact The columns of A are linearly independent

$\Updownarrow$

The equation $A\vec{x} = \vec{0}$ has only the trivial sol^{n}.

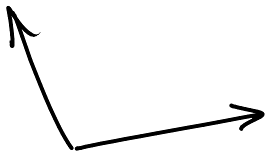
Ex Are the columns of $\begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$ lin. indep. ?

$$\left[\begin{array}{ccc|c} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{array}\right]$$

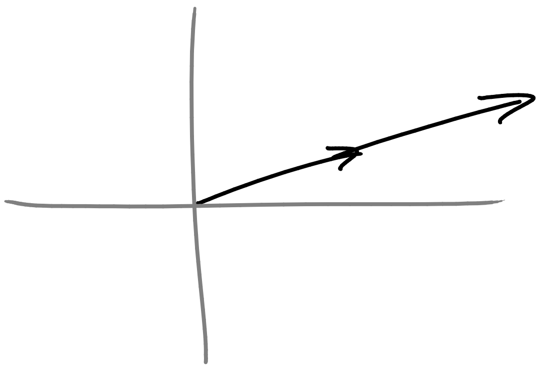
No free vars $\Rightarrow$ only the trivial solution

$\Rightarrow$ the columns are lin. indep.

Remark: Another way of saying $\vec{v}_1, \dots, \vec{v}_n$ are linearly dependent is that one of $\vec{v}_1, \dots, \vec{v}_n$ is a linear combination of the others.



lin. indep.



lin. dep.

lin. dep.

<u>Fact</u> A set of <u>one</u> vector $\vec{v}$ is linearly dep. if and only if $\vec{v} = \vec{0}$. $(1 \cdot \vec{0} = \vec{0})$

<u>Fact</u> A set of <u>two</u> vectors $\vec{v}_1, \vec{v}_2$ is linearly dep. if and only if one of them is a scalar multiple of the other.

<u>Ex</u>

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix} \right\}$ lin. dep.

$\left\{ \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ 5 \end{bmatrix} \right\}$ lin. indep.

$\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$ lin. dep.

(Don't need row reduction for these!)

Fact: Any set $\{\vec{v}_1, \dots, \vec{v}_p\}$ of vectors in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Ex 3 vectors in $\mathbb{R}^2$ are always lin. dep.
4 vectors in $\mathbb{R}^3$ " " " "
12 vectors in $\mathbb{R}^8$ " " " "

Why? To see if they're lin dep we look at

$$\begin{array}{c} n \\ \text{entries} \end{array} \updownarrow \left[\begin{array}{cccc|c} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_p & \vec{0} \end{array}\right]$$

At most one pivot in each row. So # of pivots is $\leq n$.
But # of columns is $p > n$. So there must be a column w/no pivot.
i.e. must be a free variable.
i.e. there is a nontrivial sol^n to the corresponding lin. sys.
i.e. the vectors are linearly dependent!

Fact Any set of vectors including $\vec{0}$ is lin dep.

Why? Because if we have the vectors $\{\vec{0}, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_p\}$
we can write
$$1 \cdot \vec{0} + 0\vec{v}_2 + 0\vec{v}_3 + \cdots + 0\vec{v}_p = \vec{0}$$

Fact If one of $\vec{v}_1, \ldots, \vec{v}_p$ is a multiple of another of $\vec{v}_1, \ldots, \vec{v}_p$ then $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is lin dep.

Why? If $\vec{v}_1 = c\vec{v}_2$

then $\vec{v}_1 - c\vec{v}_2 + 0\vec{v}_3 + 0\vec{v}_4 + \cdots + 0\vec{v}_p = \vec{0}$