Lecture 5

9 Sep 2010

Last time: <u>linear independence</u>

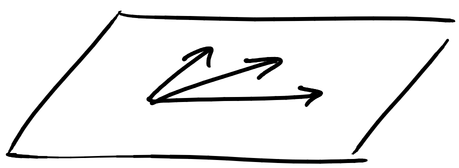
We said $\{\vec{v}_1, \ldots, \vec{v}_n\}$ are lin. indep. <u>if</u> (and only if) the <u>vector equation</u>

$$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$$

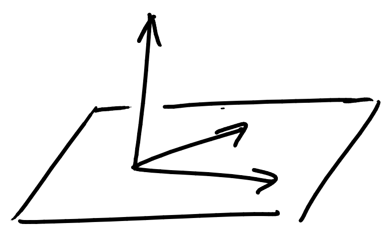
has only the <u>trivial</u> solution (all $x_i = 0$).

<u>Or</u>: $\{\vec{v}_1, \ldots, \vec{v}_n\}$ are lin indep if no one of them is a linear combination of the others

In $\mathbb{R}^3$:



lin. dep.



lin. indep.

<u>Fact</u>: If $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly dependent then so is any set of vectors <u>containing</u> $\{\vec{v}_1, \ldots, \vec{v}_k\}$

<u>Ex</u>

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 12 \\ -6 \end{bmatrix} \right\} \text{ is lin dep}$$

<u>So</u>,

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 9 \\ 11 \\ 8 \\ 4 \end{bmatrix}, \begin{bmatrix} 7 \\ 3 \\ -6 \\ -100 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 12 \\ -6 \end{bmatrix} \right\} \text{ is also lin dep.}$$

Introduction to Linear Transformations (Sec 1.8, 1.9)

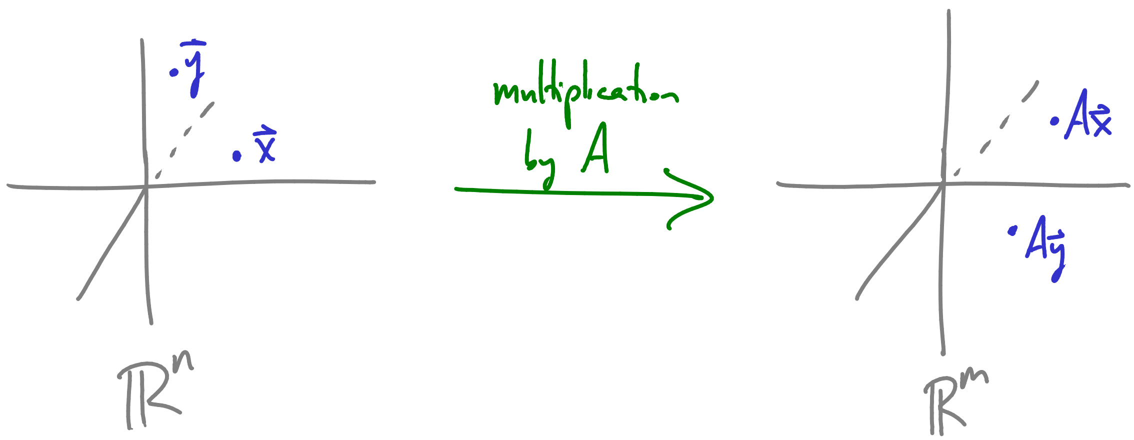
So far we looked at $A\vec{x}=\vec{b}$ as just another way of writing <u>linear equations</u>.

Now change perspective.

A $m\times n$ matrix

$\vec{x}\in\mathbb{R}^n$

$A\vec{x}\in\mathbb{R}^m$



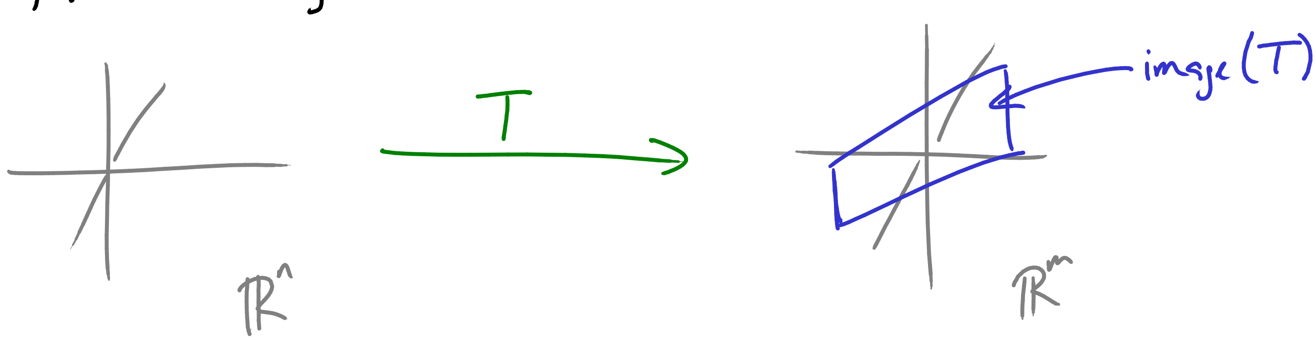
This operation is a <u>function</u> mapping $\mathbb{R}^n\to\mathbb{R}^m$

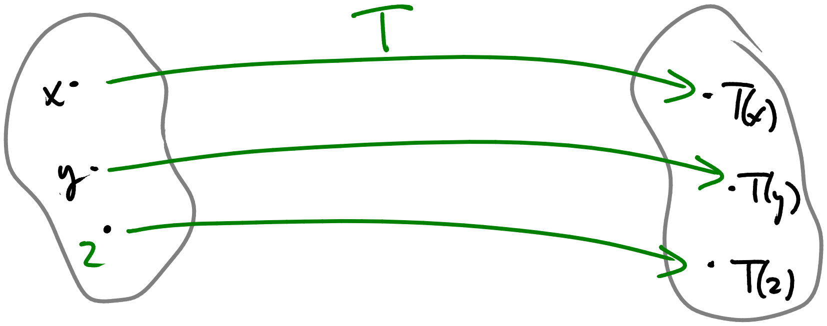
$$T:\mathbb{R}^n\to\mathbb{R}^m \qquad T(\vec{x})=A\vec{x}$$

Can ask the standard questions about functions:

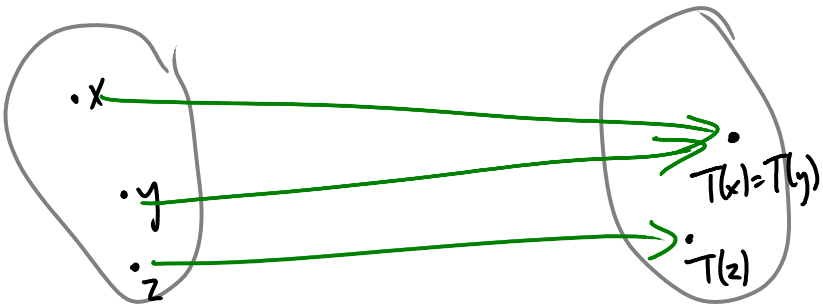
e.g. 1) is it 1–1?

2) what is its range?





we call T 1-1 if $\left[T(x) = T(y) \iff x = y \right]$.



not 1-1:
$T(x) = T(y)$
although $x \neq y$.

Ex

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \qquad \vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

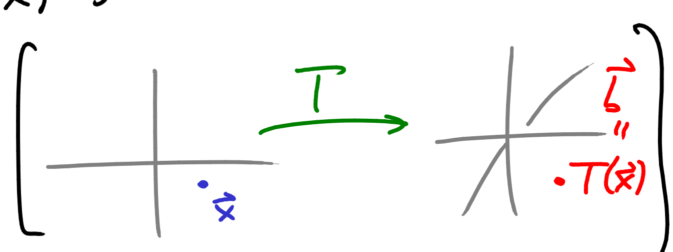
Define $T(\vec{x}) = A\vec{x}$. $\qquad T : \mathbb{R}^2 \to \mathbb{R}^3$

1) What is $T(\vec{u})$?

$$T(\vec{u}) = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}.$$

2) Say $\vec{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$. Find an $\vec{x}$ whose image under T is $\vec{b}$.

i.e. $T(\vec{x}) = \vec{b}$



i.e. solve $A\vec{x} = \vec{b}$

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{array}\right] \quad \text{i.e.} \quad \begin{array}{l} x_1 = 3/2 \\ x_2 = -1/2 \end{array}$$

$$\text{i.e.} \quad \vec{x} = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}$$

3) Is there more than one $\vec{x}$ with $T(\vec{x}) = \vec{b}$?

No: the solution we just found was unique, $\vec{x} = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}$.

4) Is $\vec{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ in the range of T?

Try to solve $T(\vec{x}) = \vec{c}$:

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 25 \end{array}\right]$$

Inconsistent $\Longrightarrow$ no solution for $\vec{x}$

$\Longrightarrow$ $\vec{c}$ is not in the range of T.

We just studied the function $T(\vec{x}) = A\vec{x}$ for some matrix A. What are the properties that every such T has?

I) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$

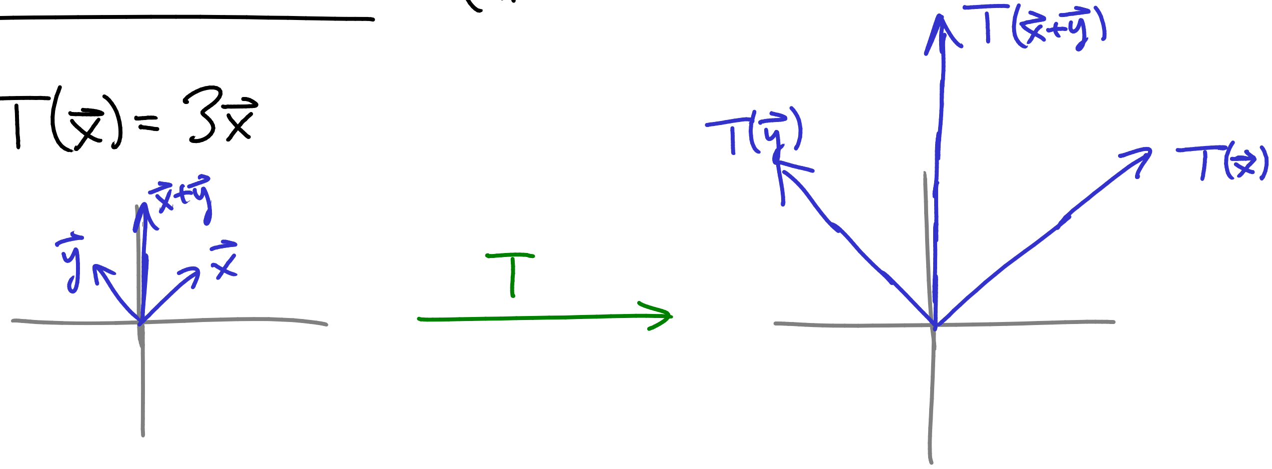
[Why? $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$]

II) $T(c\vec{v}) = cT(\vec{v})$

Any function $T: \mathbb{R}^n \to \mathbb{R}^m$ with these properties I, II is called a <u>linear transformation</u>.

<u>Examples of linear transformations</u> $(\mathbb{R}^2 \to \mathbb{R}^2)$

1) $T(\vec{x}) = 3\vec{x}$



Why is T linear? $T(\vec{x}+\vec{y}) \stackrel{?}{=} T(\vec{x}) + T(\vec{y})$

$$3(\vec{x}+\vec{y}) = 3(\vec{x}) + 3(\vec{y}) \quad \checkmark$$

$$T(c\vec{x}) \stackrel{?}{=} cT(\vec{x})$$

$$3(c\vec{x}) = c(3\vec{x}) \quad \checkmark$$

<u>Is T 1-1?</u>

Suppose $T(\vec{u}) = T(\vec{v})$.

$$3\vec{u} = 3\vec{v}$$

mult. both sides by $\frac{1}{3}$: $\frac{1}{3}\cdot 3\vec{u} = \frac{1}{3}\cdot 3\vec{v}$

$$\vec{u} = \vec{v}$$

So $T(\vec{u}) = T(\vec{v}) \Rightarrow \vec{u} = \vec{v}$, so <u>T is 1-1</u> $\checkmark$.

<u>What is the range of T</u>?

i.e. what is the set of vectors $\vec{z}$ such that $\vec{z} = T(\vec{x})$ for some $\vec{x}$?

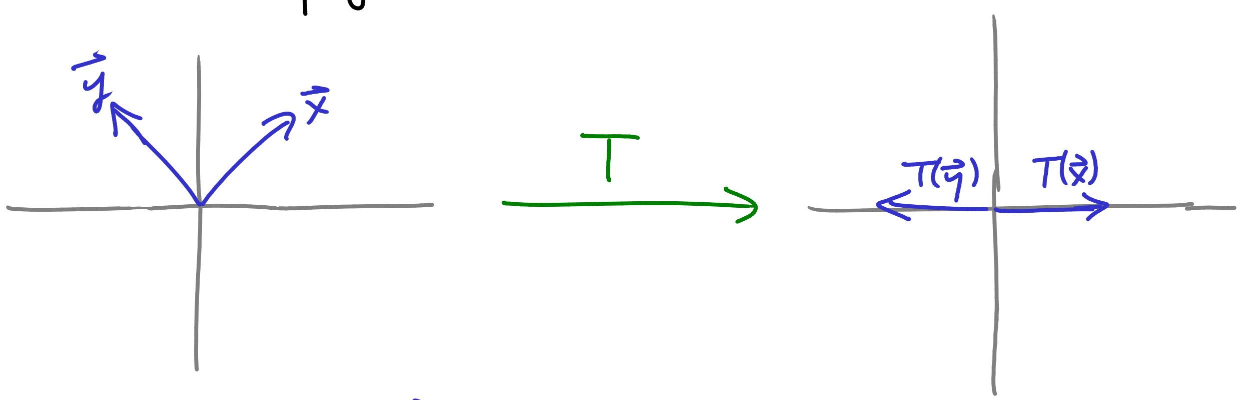
For any vector $\vec{z}$, we want to solve

$$\vec{z} = T(\vec{x})$$
$$\vec{z} = 3\vec{x}$$
$$\tfrac{1}{3}\vec{z} = \vec{x}$$

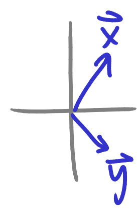
So $\vec{x} = \frac{1}{3}\vec{z}$ has $T(\vec{x}) = \vec{z}$, for any $\vec{z}$

So the range of T is the set of all $\vec{z} \in \mathbb{R}^2$.

2) $T(\vec{x})$ = projection of $\vec{x}$ to the x-axis



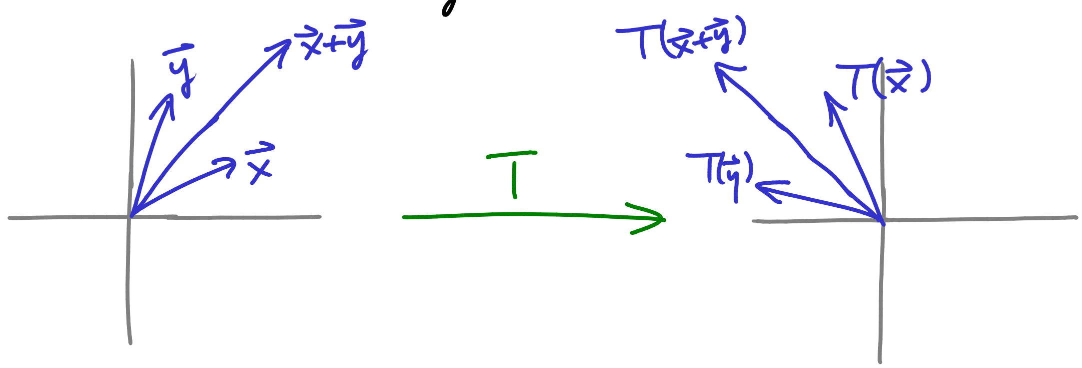
T is <u>Not</u> 1-1



$T(\vec{x}) = T(\vec{y})$

The range of T is the x-axis (not all of $\mathbb{R}^2$)

3) $T(\vec{x})$ = rotation of $\vec{x}$ by 90° counterclockwise



T is 1-1

The range of T is all of $\mathbb{R}^2$

Fact: Linear transformations obey

$$T(\vec{0}) = \vec{0}$$

$$\left[\text{Why? Because } T(\vec{0}) = T(0\cdot\vec{v}) = 0\cdot T(\vec{v}) = \vec{0}\right]$$

Fact: Lin. trans. obey

$$T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + d\,T(\vec{v})$$

and even

$$T(c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n) = c_1T(\vec{u}_1) + c_2T(\vec{u}_2) + \cdots + c_nT(\vec{u}_n)$$

So in ptic: let's look again at linear maps $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^n$

Suppose we know $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \vec{a}_1 \qquad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \vec{a}_2$$

Then we can figure out

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = T\left(x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= x_1\,T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + x_2\,T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= x_1\vec{a}_1 + x_2\vec{a}_2$$

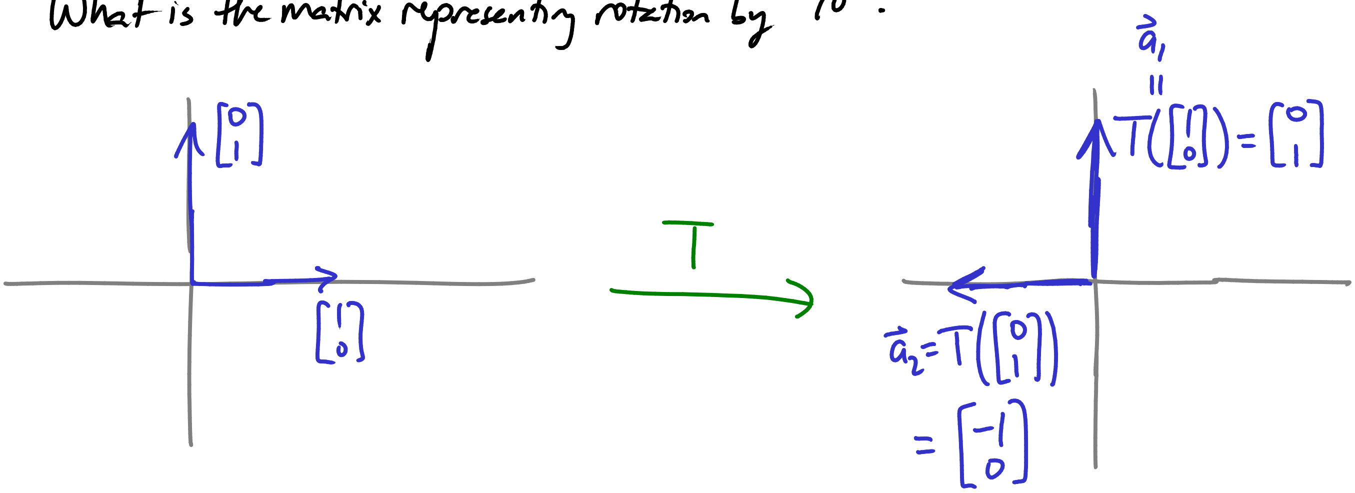
In other words: $T(\vec{x}) = A \cdot \vec{x}$

where $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix}$

(ex if $\vec{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$, then $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 4 & -2 \end{bmatrix}$)

A is "the matrix representing the linear transformation T"

<u>Ex</u> What is the matrix representing rotation by $90°$?



So the matrix rep. T is $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(Similarly, a rotation by an angle θ is represented by the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$)