Theorem.
Let \( f \) be a real-valued function. Let \( D \subset \mathbb{R} \) be the domain of \( f \), and \( p \in D \). Then,

\[ f \text{ is continuous at } p \]

if and only if

for all sequences \( (x_n) \subset D \) such that \( x_n \to p \), we have \( f(x_n) \to f(p) \).

Proof.
First, we prove the forward direction: assume that \( f \) is continuous at \( p \), and suppose given some sequence \( (x_n) \subset D \), such that \( x_n \to p \). We would like to show that \( f(x_n) \to f(p) \).

Fix some arbitrary \( \epsilon > 0 \). Since \( f \) is continuous at \( p \), there exists a \( \delta > 0 \) such that

\[
(x \in D, |x - p| < \delta) \implies |f(x) - f(p)| < \epsilon.
\]

Also, since \( x_n \to p \), there exists an \( N \in \mathbb{N} \) such that

\[
n \geq N \implies |x_n - p| < \delta.
\]

Combining these two (and the fact that \( x_n \in D \) from above), we have that

\[
n \geq N \implies |f(x_n) - f(p)| < \epsilon.
\]

So \( f(x_n) \to f(p) \).

Next, we prove the backward direction. For this we switch to its contrapositive. So, assume that \( f \) is not continuous at \( p \). We would like to show that there exists some sequence \( (x_n) \subset D \), such that \( x_n \to p \), and \( f(x_n) \not\to f(p) \).

Since \( f \) is not continuous at \( p \), there exists some \( \epsilon > 0 \) such that, for all \( \delta > 0 \), there exists an \( x \in D \) with \( |x - p| < \delta \) and \( |f(x) - f(p)| \geq \epsilon \). Fix this \( \epsilon \). Then for any \( n \in \mathbb{N} \), taking \( \delta = 1/n \), it follows that there exists an \( x_n \in D \) with \( |x_n - p| < 1/n \) and \( |f(x_n) - f(p)| \geq \epsilon \).

This defines our sequence \( (x_n) \subset D \).

Since \( |x_n - p| < 1/n \), we have \( p - 1/n \leq x_n \leq p + 1/n \); and \( p + 1/n \to p \), \( p - 1/n \to p \), so applying the “Squeeze Theorem” (problem 3.19) gives \( x_n \to p \).

But since \( |f(x_n) - f(p)| \geq \epsilon \) for all \( n \), \( f(x_n) \not\to f(p) \) (problem 3.10).

So we have shown that \( (x_n) \) has all the desired properties.