In your solutions to these exercises you may freely use any results proven in class or in Rudin chapters 1-6, without reproving them.

**Exercise 1** *(Rudin 6.2)*

Suppose $f(x) \geq 0$ for all $x \in [a,b]$, $f$ is continuous, and $\int_a^b f(x) \, dx = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$.

**Answer of exercise 1**

Suppose for contradiction that $f(y) = c \neq 0$ at some $y \in [a,b]$. Then by continuity, there exists some neighborhood $N_\epsilon(y)$ such that $f(x) > c/2$ for all $x \in N$. Let $m$ be the infimum of $f(x)$ for $x \in I$; then $m \geq c/2$. The full lower sum $L(P, f)$ is obtained by summing the contribution from the interval $I$ plus the contributions from other intervals. All those contributions are nonnegative, so $L(P, f)$ is at least the contribution from $I$, i.e. $L(P, f) \geq m \epsilon$. But then

$$\int_a^b f(x) \, dx \geq L(P, f) \geq m \epsilon > 0.$$  

**Exercise 2** *(Rudin 6.5)*

Suppose $f$ is a bounded real function on $[a,b]$ and $f^2$ is Riemann integrable on $[a,b]$. Does it follow that $f$ is Riemann integrable on $[a,b]$? Does the answer change if we assume instead that $f^3$ is Riemann integrable on $[a,b]$?

**Answer of exercise 2**

If $f^2$ is Riemann integrable it need not follow that $f$ is; a counterexample is provided by the function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ -1 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

However, if $f^3$ is Riemann integrable then the situation is better. Indeed, for any $x$ we can define a “cube root” $x^{1/3}$, such that $(x^3)^{1/3} = x$. (We had defined $x^{1/3}$ before only for $x \geq 0$; but we can extend it to $x < 0$ by defining $x^{1/3} = -|x|^{1/3}$ for $x < 0$. Then we can check directly that the resulting function indeed has $(x^3)^{1/3} = x$ for all $x$.) Moreover this function is continuous (we have proved before that it is continuous for $x \geq 0$, but this easily implies it is continuous for all $x$.) Then $f(x) = (f^3)^{1/3}$, and $f^3$ is integrable, so $f$ is obtained by applying a continuous function to an integrable function, so $f$ is also integrable.

**Exercise 3** *(Rudin 6.7, in part)*
Suppose \( f \) is a real function on \((0, 1]\) and \( f \) is Riemann integrable on \([c, 1]\) for every \( c > 0 \). We then define
\[
\int_0^1 f(x) \, dx = \lim_{c \to 0} \int_c^1 f(x) \, dx
\]
if this limit exists.
If \( f \) is Riemann integrable on \([0, 1]\), show that this definition agrees with the old one.

**Answer of exercise 3**

The easy way: if \( f \) is Riemann integrable then the function \( F(c) = \int_c^1 f(x) \, dx \) is **continuous** on \([0, 1]\) (using Rudin’s Theorem 6.20). Thus
\[
\lim_{c \to 0} F(c) = F(0)
\]
which means
\[
\lim_{c \to 0} \int_c^1 f(x) \, dx = \int_0^1 f(x) \, dx
\]
which is what we wanted to prove.

The harder way (doing it “by hand”): if \( f \) is Riemann integrable on \([0, 1]\) then in particular it is bounded, say \( |f(x)| < M \) for all \( x \in [0, 1] \). Thus
\[
\left| \int_0^c f(x) \, dx \right| \leq \int_0^c |f(x)| \, dx \leq Mc
\]
so
\[
0 \leq \lim_{c \to 0} \left| \int_0^c f(x) \, dx \right| \leq \lim_{c \to 0} \int_0^c |f(x)| \, dx \leq \lim_{c \to 0} Mc = 0
\]
and hence
\[
\lim_{c \to 0} \left| \int_0^c f(x) \, dx \right| = 0
\]
which is equivalent to
\[
\lim_{c \to 0} \int_0^c f(x) \, dx = 0.
\]

Now
\[
\int_0^1 f(x) \, dx = \int_0^1 f(x) \, dx - \int_c^0 f(x) \, dx
\]
and so
\[
\lim_{c \to 0} \int_0^1 f(x) \, dx = \int_0^1 f(x) \, dx - \lim_{c \to 0} \int_c^0 f(x) \, dx
\]
i.e.
\[
\lim_{c \to 0} \int_c^1 f(x) \, dx = \int_0^1 f(x) \, dx
\]
as desired.